Equilibrated error estimators for magnetostatic problems

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Motivation

We will be concerned with PDE problems in a domain $\Omega \subset \mathbb{R}^3$ with solution u.

We will compute a FE approximation u_h, build on a mesh $\mathcal{T}_h$.

Our ideal goal is to compute quantities η_K called "estimators" such that

$$\eta_K \simeq \|u - u_h\|_K$$

for all $K \in \mathcal{T}_h$.

These "error estimators" are useful to
(i) reliably assess the error, and
(ii) drive adaptive refinement processes.

Reliability and efficiency

Efficiency

$$\eta_K \leq C_{\text{eff}} \, \|u - u_h\|_{\widetilde{K}}$$

Reliability

$$\|u - u_h\|_{\Omega} \leq C_{\text{rel}} \eta, \quad \eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2.$$

The goal is to design η_K such that C_{eff} and C_{rel} are as "nice" as possible.

Why flux equilibration?

Equilibrated estimators are one of the many families of estimators available.

With "traditional" estimators, we have $C_{\rm rel} = C_{\rm rel}(\kappa_{\mathcal{T}_h}, p)$, $C_{\rm eff} = C_{\rm eff}(\kappa_{\mathcal{T}_h}, p)$.

With equilibrated estimators, we have $C_{\rm rel} = 1$ and $C_{\rm eff} = C_{\rm eff}(\kappa_{\mathcal{T}_h})$.

In words, we have a "guaranteed upper bound" and which is "p-robust".

A quick literature review before actually starting

Flux equilibration for scalar problems:

- Prager and Synge, 1947: the initial idea
- Destuynder and Métivet, 1999: efficient flux construction
- Braess, Pillwein and Schöberl, 2009: p-robustness in 2D
- Ern and Vohralík, 2020: p-robustness in 3D

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- Ern and Vohralík, 2020: *p*-robustness in 3D

Flux equilibration for magnetostatic problems:

- Braess and Schöberl, 2008: lowest–order case
- Gedicke, Geevers and Perugia, 2020: arbitrary–order
- Gedicke, Geevers, Perugia and Schöberl, 2021: *p*-robustness
- Chaumont-Frelet and Vohralík, 2021: today's menu

Outline

Equilibration for the Poisson problem

Equilibration for the Poisson problem

A brief physical discussion

The Poisson problem

Well, mathematically, it's just: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \rho & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\rho : \Omega \to \mathbb{R}$ is given.

For reasons that will become clear soon, I'd like to give a "physical" context.

Consider a vacuum Ω enclosed by a perfect conductor.
Choose units such that $\varepsilon_0 = 1$.
Consider a charge density $\rho : \Omega \to \mathbb{R}$.

We want to model the electric field $\boldsymbol{E} : \Omega \to \mathbb{R}^3$.

Poisson = Gauss + Faraday

Gauss' law (with $\varepsilon_0 = 1$):

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \text{ in } \Omega.$$

Faraday's law (with $\partial_t \boldsymbol{B} = \boldsymbol{o}$):

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{o} \text{ in } \Omega, \qquad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{o} \text{ on } \partial\Omega.$$

Assuming Ω is simply-connected we can rephrase Faraday's law:
There exists an electric potential u, with $u = 0$ on $\partial\Omega$ such that

$$\boldsymbol{E} = -\boldsymbol{\nabla} u$$

Combining both laws, we get back where we started:

$$\begin{cases} -\Delta u = \rho & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Equilibration for the Poisson problem

Discretization and main ideas

Variational formulation and discretization

Assuming that $\rho \in L^2(\Omega)$, we seek $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (\rho, v)_\Omega$$

for all $v \in H_0^1(\Omega)$.

Given $p \geq 0$, we consider the conforming Lagrange FE space

$$V_h := \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$$

where $\mathcal{T}_h$ is a simplicial mesh of Ω.

We compute the discrete approximation $u_h \in V_h$ by solving

$$(\nabla u_h, \nabla v_h)_\Omega = (\rho, v_h)_\Omega$$

for all $v_h \in V_h$.

Which physical laws are satisfied?

Let us call $\widetilde{\boldsymbol{E}}_h := -\boldsymbol{\nabla} u_h$ the electric field approximation.

By construction, $\widetilde{\boldsymbol{E}}_h$ does satisfy Faraday's law:
it is the gradient of the (conforming) potential $u_h \in H^1_0(\Omega)$.

On the other hand, $\widetilde{\boldsymbol{E}}_h$ violates Gauss' law. We do not have

$$\text{“}\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{E}}_h = \rho\text{”},$$

and actually, it is not even true that “$\widetilde{\boldsymbol{E}}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$”.

$\boldsymbol{E}$ being the unique field satisfying both laws,
“measuring” how Gauss' law is violated by $\widetilde{\boldsymbol{E}}_h$ provides a bound on

$$\|\boldsymbol{E} - \widetilde{\boldsymbol{E}}_h\|_\Omega = \|\boldsymbol{\nabla}(u - u_h)\|_\Omega.$$

How do we "measure" Gauss' law violation?

We call "equilibrated flux" any field $\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\boldsymbol{\nabla} \cdot \boldsymbol{E}_h = \rho$.

Equilibrated fluxes are those fields satisfying Gauss' law.

Consider the minimization problem

$$\min_{\substack{\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla} \cdot \boldsymbol{E}_h = \rho}} \|\boldsymbol{E}_h + \boldsymbol{\nabla} u_h\|_\Omega.$$

The minimum is zero if and only if the error is zero.
Indeed, then $\boldsymbol{E}_h = -\boldsymbol{\nabla} u_h =: \widetilde{\boldsymbol{E}}_h$ satisfies Gauss' law.

We want to build a flux $\boldsymbol{E}_h$ (i.e. a field satisfying Gauss' law) close to $-\boldsymbol{\nabla} u_h$.

Equilibration for the Poisson problem

The Prager–Synge inequality

The Prager–Synge inequality

Equilibrated flux

$$\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}, \Omega) \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E}_h = \rho$$

Prager–Synge inequality

$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \leq \|\boldsymbol{E}_h + \boldsymbol{\nabla} u_h\|_\Omega$$

Saturation of the bound

$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega = \|(-\boldsymbol{\nabla} u) + \boldsymbol{\nabla} u_h\|_\Omega$$

The proof is easy!

Consider an equilibrated flux $\boldsymbol{E}_h$ and any $v \in H_0^1(\Omega)$. We have

$$\begin{aligned}(\boldsymbol{\nabla}(u - u_h), \boldsymbol{\nabla} v)_\Omega &= (\rho, v)_\Omega - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega \\ &= (\boldsymbol{\nabla} \cdot \boldsymbol{E}_h, v)_\Omega - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega \\ &= -(\boldsymbol{E}_h + \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega.\end{aligned}$$

Picking $v = u - u_h$, the result follows:

$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \leq \|\boldsymbol{E}_h + \boldsymbol{\nabla} u_h\|_\Omega$$

for all equilibrated fluxes $\boldsymbol{E}_h$.

Equilibration for the Poisson problem

Practical construction of discrete fluxes

Piecewise polynomial right-hand sides

In the remainder of the talk, we will assume for simplicity that $\rho \in \mathcal{P}_p(\mathcal{T}_h)$.

General rhs can be considered, by adding “oscillation” terms in the estimator.

This assumption allows the construction of discrete fluxes $\boldsymbol{E}_h$.

Raviart–Thomas elements

We want to construct discrete functions $\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\boldsymbol{\nabla} \cdot \boldsymbol{E}_h \in \mathcal{P}_p(\mathcal{T}_h)$.

The "correct" tool for this is the Raviart–Thomas FE space

$$\boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega).$$

The following key property holds true:

$$\boldsymbol{\nabla} \cdot [\boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)] = \mathcal{P}_p(\mathcal{T}_h).$$

There exist discrete fluxes $\boldsymbol{E}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\boldsymbol{\nabla} \cdot \boldsymbol{E}_h = \rho$.

The "ideal" discrete flux

The ideal discrete flux is the one minimizing the upper bound, i.e.,

$$\boldsymbol{E}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = \rho}} \|\boldsymbol{v}_h + \boldsymbol{\nabla} u_h\|_\Omega.$$

It is fully computable: find $\boldsymbol{E}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ and $q \in \mathcal{P}_p(\mathcal{T}_h)$ s.t.

$$\left\{ \begin{array}{rcll} (\boldsymbol{E}_h, \boldsymbol{v}_h)_\Omega + (q_h, \boldsymbol{\nabla} \cdot \boldsymbol{v}_h) & = & -(\boldsymbol{\nabla} u_h, \boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega), \\ (\boldsymbol{\nabla} \cdot \boldsymbol{E}_h, w_h) & = & (\rho, w_h) & \forall w_h \in \mathcal{P}_p(\mathcal{T}_h). \end{array} \right.$$

There are, however, two issues:
(i) the above problem is more expensive than the one we started with, and
(ii) it is not clear that we will obtain local informations.

Equilibration for the Poisson problem

Localization

Localization

We have the "ideal" discrete definition

$$\boldsymbol{E}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = \rho}} \|\boldsymbol{v}_h + \boldsymbol{\nabla} u_h\|_\Omega.$$

We can observe that it is the discrete version of

$$\boldsymbol{E} := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = \rho}} \|\boldsymbol{v} + \boldsymbol{\nabla} u\|_\Omega,$$

for which $\boldsymbol{E} = -\boldsymbol{\nabla} u$.

The definition $\boldsymbol{E}_h$ thus mimics the one of $\boldsymbol{E}$ as a minimizer.

The idea is then to decompose $\boldsymbol{E}$ into local contributions $\boldsymbol{E}^{\boldsymbol{a}}$, and to define local contributions $\boldsymbol{E}_h^{\boldsymbol{a}}$ by mimicking $\boldsymbol{E}^{\boldsymbol{a}}$.

Partition of unity

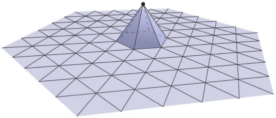
Let $\mathcal{V}_h$ denotes the set of vertices of the mesh $\mathcal{T}_h$.
Let $\psi^{\boldsymbol{a}}$ denote the "hat function" associated with $\boldsymbol{a} \in \mathcal{V}_h$.
Then:

(i) we always have have $\psi^{\boldsymbol{a}} \in V_h$ for all $p \geq 0$,

(ii) $\psi^{\boldsymbol{a}}$ is supported in the patch $\omega^{\boldsymbol{a}}$ of elements $K \in \mathcal{T}_h$ sharing the vertex $\boldsymbol{a}$,

(iii) we have the partition of unity property

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} = 1.$$



Localization of the continuous flux

Consider $\boldsymbol{E}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \boldsymbol{E}$. Then, we have

$$\boldsymbol{E} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{E}^{\boldsymbol{a}}.$$

In addition, we actually have $\boldsymbol{E}^{\boldsymbol{a}} = -\psi^{\boldsymbol{a}} \boldsymbol{\nabla} u \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}})$, and

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u.$$

We thus have the characterization

$$\boldsymbol{E}^{\boldsymbol{a}} = \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u}} \|\boldsymbol{v} + \psi^{\boldsymbol{a}} \boldsymbol{\nabla} u\|_{\omega^{\boldsymbol{a}}}.$$

At boundary nodes $\boldsymbol{a} \in \mathcal{V}_h$ these definitions are slightly adapted.

Local construction of a discrete flux

At the continuous level, we arrived at

$$\boldsymbol{E}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u}} \|\boldsymbol{v} + \psi^{\boldsymbol{a}} \boldsymbol{\nabla} u\|_{\omega^{\boldsymbol{a}}}.$$

This motivates the discrete definition

$$\boldsymbol{E}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h}} \|\boldsymbol{v}_h + \psi^{\boldsymbol{a}} \boldsymbol{\nabla} u_h\|_{\omega^{\boldsymbol{a}}}.$$

Does this definition make sense?

The magic of Galerkin orthogonality

Recall the minimization problem

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h}} \|\boldsymbol{v}_h + \psi^{\boldsymbol{a}} \boldsymbol{\nabla} u_h\|_{\omega^{\boldsymbol{a}}}.$$

Stokes' formula implies that the compatibility condition

$$\int_{\omega^{\boldsymbol{a}}} \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h = 0$$

must be satisfied! Because $\psi^{\boldsymbol{a}} \in V_h$, it "magically" works, since

$$0 = (\rho, \psi^{\boldsymbol{a}})_\Omega - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} \psi^{\boldsymbol{a}})_\Omega = \int_{\omega^{\boldsymbol{a}}} \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h.$$

The degree of the polynomials is also properly chosen.

Gluing the pieces together

We have properly defined local contributions

$$\boldsymbol{E}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h}} \|\boldsymbol{v}_h + \psi^{\boldsymbol{a}} \boldsymbol{\nabla} u_h\|_{\omega^{\boldsymbol{a}}}.$$

Setting

$$\boldsymbol{E}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{E}_h^{\boldsymbol{a}},$$

we have $\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}, \omega^{\boldsymbol{a}})$ thanks to the b.c. of each $\boldsymbol{E}_h^{\boldsymbol{a}}$, and

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E}_h &= \sum_{\boldsymbol{a} \in \mathcal{V}_h} (\psi^{\boldsymbol{a}} \rho - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h) \\ &= \left(\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} \right) \rho - \boldsymbol{\nabla} \left(\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} \right) \cdot \boldsymbol{\nabla} u_h \\ &= \rho. \end{aligned}$$

We have successful defined an equilibrated flux from local prescriptions!

Summary of the construction

(i) Compute the discrete solution $u_h \in V_h$ by solving the linear system

$$(\nabla u_h, \nabla v_h)_\Omega = (\rho, v_h)_\Omega \quad \forall v_h \in V_h.$$

(ii) For each $\mathbf{a} \in \mathcal{V}_h$, compute the local field contribution

$$\boldsymbol{E}_h^{\mathbf{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\mathbf{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \psi^{\mathbf{a}} \rho - \nabla \psi^{\mathbf{a}} \cdot \nabla u_h}} \|\boldsymbol{v}_h + \psi^{\mathbf{a}} \nabla u_h\|_\Omega$$

by solving small uncoupled linear systems with Lagrange multipliers.

(iii) Assemble the local contributions

$$\boldsymbol{E}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{E}_h^{\mathbf{a}}$$

(iv) $\boldsymbol{E}_h$ is an equilibrated flux, so that we have

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{E}_h + \nabla u_h\|_\Omega.$$

Equilibration for the Poisson problem

Efficiency

Efficiency

Our estimator is

$$\eta := \|\boldsymbol{E}_h + \boldsymbol{\nabla} u_h\|_\Omega,$$

and we may associate with element $K \in \mathcal{T}_h$

$$\eta_K := \|\boldsymbol{E}_h + \boldsymbol{\nabla} u_h\|_K.$$

Then, it is possible to show that

$$\eta_K \leq C_{\mathrm{eff}} \|\boldsymbol{\nabla}(u - u_h)\|_{\widetilde{K}},$$

with a constant C_{eff} independent of p.

Unfortunately, I have no time for the proofs, which are technical.

Equilibration for the Poisson problem

Takeaways

Takeaways

Equilibrated fluxes provide guaranteed bounds via the Prager–Synge inequality.

For the Poisson problem, Raviart–Thomas elements are natural for the flux.

The flux may be computed by solving local mixed FE problems.

For this particular flux construction, we have p-robust efficiency estimates.

Equilibration for the magnetostatic problem

Equilibration for the magnetostatic problem

Another brief physical interpretation

The magnetostatic problem

Mathematically, the problem consists in finding $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ such that

$$\left\{ \begin{array}{rcll} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} & = & \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{\nabla} \cdot \boldsymbol{A} & = & 0 & \text{in } \Omega, \\ \boldsymbol{A} \times \boldsymbol{n} & = & \boldsymbol{o} & \text{on } \partial\Omega, \end{array} \right.$$

where $\boldsymbol{J} : \Omega \to \mathbb{R}^3$ is a given rhs s.t. $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$.

As before, it is made out of two physical laws.

We still assume that Ω is a vacuum enclosed by a conductor.
We choose units such that $\mu_0 = 1$.
$\boldsymbol{J}$ is now a "current density".

What we want to model is the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$.

The two physical laws

Ampere's law:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \text{ in } \Omega.$$

Absence of magnetic monopoles (with $\mu_0 = 1$):

$$\nabla \cdot \boldsymbol{H} = 0 \text{ in } \Omega, \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega.$$

There exists a "magnetic potential" $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ s.t.

$$\nabla \times \boldsymbol{A} = \boldsymbol{H} \text{ in } \Omega \qquad \boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{o} \text{ on } \partial\Omega.$$

We then arrive at the "curl–curl" problem:

$$\left\{ \begin{array}{rcll} \nabla \times \nabla \times \boldsymbol{A} & = & \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{A} \times \boldsymbol{n} & = & \boldsymbol{o} & \text{on } \partial\Omega. \end{array} \right.$$

Discretization and main ideas

Variational formulation and discretization

Assuming that $\boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}^0, \Omega)$, we seek $\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and $q \in H_0^1(\Omega)$ s.t.

$$\left\{ \begin{array}{rcll} (\boldsymbol{\nabla} \times \boldsymbol{A}, \boldsymbol{\nabla} \times \boldsymbol{v})_\Omega + (\boldsymbol{\nabla} q, \boldsymbol{v})_\Omega & = & (\boldsymbol{J}, \boldsymbol{v})_\Omega & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega), \\ (\boldsymbol{A}, \boldsymbol{\nabla} w)_\Omega & = & 0 & \forall w \in H_0^1(\Omega). \end{array} \right.$$

For the discretization we introduce the Nédélec FE space

$$\boldsymbol{W}_h := \boldsymbol{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

The discrete formulation is to seek $\boldsymbol{A}_h \in \boldsymbol{W}_h$ and $q_h \in V_h$ s.t.

$$\left\{ \begin{array}{rcll} (\boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{\nabla} \times \boldsymbol{v}_h)_\Omega + (\boldsymbol{\nabla} q_h, \boldsymbol{v}_h)_\Omega & = & (\boldsymbol{J}, \boldsymbol{v}_h)_\Omega & \forall \boldsymbol{v} \in \boldsymbol{W}_h, \\ (\boldsymbol{A}_h, \boldsymbol{\nabla} w_h)_\Omega & = & 0 & \forall w_h \in V_h. \end{array} \right.$$

Removing the Lagrange multipliers

We can show that $q = 0$ and $q_h = 0$, so that we actually have

$$(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v})_\Omega = (\boldsymbol{J}, \boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega),$$

and

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h)_\Omega = (\boldsymbol{J}, \boldsymbol{v}_h)_\Omega \qquad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h,$$

which is the "$\boldsymbol{H}(\mathbf{curl})$ version" of the $(\nabla\cdot, \nabla\cdot)_\Omega$ problem.

Which laws are actually satisfied?

Let us call $\widetilde{\boldsymbol{H}}_h := \boldsymbol{\nabla} \times \boldsymbol{A}_h$ the magnetic field approximation.

Then by construction, $\widetilde{\boldsymbol{H}}_h$ satisfies the "no magnetic monopoles" law: it is the curl of a (conforming) magnetic potential $\boldsymbol{A}_h \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$.

On the other hand, it violates Ampere's law, and we do not have

$$\text{``}\boldsymbol{\nabla} \times \widetilde{\boldsymbol{H}}_h = \boldsymbol{J}\text{''}$$

and actually, it is not even true that $\widetilde{\boldsymbol{H}}_h \in \boldsymbol{H}(\mathbf{curl}, \Omega)$.

We now want to "measure" how Ampere's is violated to provide a bound on

$$\|\boldsymbol{H} - \widetilde{\boldsymbol{H}}_h\|_\Omega = \|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega.$$

Equilibration for the magnetostatic problem

The Prager–Synge inequality in $\boldsymbol{H}(\mathbf{curl})$

The Prager–Synge inequality in $\boldsymbol{H}(\mathbf{curl})$

Equilibrated flux

$$\boldsymbol{H}_h \in \boldsymbol{H}(\mathbf{curl}, \Omega) \qquad \boldsymbol{\nabla} \times \boldsymbol{H}_h = \boldsymbol{J}$$

Prager–Synge inequality

$$\|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \|\boldsymbol{H}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_\Omega$$

Saturation of the bound

$$\|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega = \|(\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_\Omega$$

Equilibration for the magnetostatic problem

Practical construction of discrete fluxes

Piecewise polynomial right-hand sides

From now on, we will assume for simplicity that $\boldsymbol{J} \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}^0, \Omega)$.

This assumption allows for discrete fluxes $\boldsymbol{H}_h \in \boldsymbol{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$.

General rhs can be considered, by adding "oscillation" terms in the estimator.

The "ideal" discrete flux

The ideal discrete flux is the one minimizing the upper bound, i.e.,

$$\boldsymbol{H}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}, \Omega) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \boldsymbol{J}}} \|\boldsymbol{v}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_\Omega.$$

It is actually fully computable, but as before, there are two issues:
(i) the above problem is more expensive than the one we started with, and
(ii) it is not clear that we will obtain local informations.

Equilibration for the magnetostatic problem

Localization

Consider $\boldsymbol{H}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \nabla \times \boldsymbol{A}$. We have

$$\nabla \times \boldsymbol{H}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \nabla \times \nabla \times \boldsymbol{A} + \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A} = \psi^{\boldsymbol{a}} \boldsymbol{J} + \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A},$$

so that

$$\boldsymbol{H}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \omega^{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v} = \psi^{\boldsymbol{a}} \boldsymbol{J} + \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}}} \|\boldsymbol{v} - \psi^{\boldsymbol{a}} \nabla \times \boldsymbol{A}\|_{\omega^{\boldsymbol{a}}}.$$

It is thus tempting to “define” a local flux contribution by

$$\boldsymbol{H}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega^{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{J} + \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h}} \|\boldsymbol{v} - \psi^{\boldsymbol{a}} \nabla \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}},$$

but is it a sound definition?

The key problem

The minimization problem

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}},$$

is “wrong”, because the minimization is empty!

The problem is that

$$\boldsymbol{\nabla} \cdot (\psi^{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h) \neq 0,$$

and actually, we do not even have

$$\text{“}\psi^{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h \in \boldsymbol{H}(\mathrm{div}, \omega^{\boldsymbol{a}})\text{”!}$$

We need to work on the $\boldsymbol{\nabla} \psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$ term.

Equilibration for the magnetostatic problem

Fixing the “bad” term

The "bad" term

At the continuous level, we would like to set $\boldsymbol{H}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \nabla \times \boldsymbol{A}$, so that

$$\boldsymbol{J}^{\boldsymbol{a}} := \nabla \times \boldsymbol{H}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \boldsymbol{J} + \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A} \in \boldsymbol{H}_0(\mathrm{div}^0, \omega^{\boldsymbol{a}}).$$

Let us set $\boldsymbol{\theta}^{\boldsymbol{a}} := \nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}$, so that $\boldsymbol{J}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\theta}^{\boldsymbol{a}}$.

This "obvious" discrete counterpart $\nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h$ is "wrong".

What are the key properties of $\boldsymbol{\theta}^{\boldsymbol{a}}$?

We want to impose that $\boldsymbol{\nabla} \times \boldsymbol{H}^{\boldsymbol{a}} = \boldsymbol{J}^{\boldsymbol{a}}$. So that we must have

$$\boldsymbol{J}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}^0, \omega^{\boldsymbol{a}}) \quad \text{and} \quad \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{J}^{\boldsymbol{a}} = \boldsymbol{J}.$$

For $\boldsymbol{\theta}^{\boldsymbol{a}}$, what it means is that

$$\boldsymbol{\theta}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{\theta}^{\boldsymbol{a}} = -\boldsymbol{\nabla} \cdot (\psi^{\boldsymbol{a}} \boldsymbol{J}) = -\boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{J}$$

and

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\theta}^{\boldsymbol{a}} = \boldsymbol{o}.$$

The goal is to “tweak” $\boldsymbol{\nabla} \psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$ to obtain these two properties!

The plan

We are going to achieve this goal in two steps.

Step 1: construct $\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}})$ such that

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J},$$

but with

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \neq \boldsymbol{o}.$$

Step 2: compute a correction $\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}})$ such that

$$\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = 0$$

and

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = \widehat{\boldsymbol{\theta}}_h.$$

Then, $\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ does the job!

Step 2: the key idea

The summed contribution from the first step necessarily satisfies $\nabla \cdot \widehat{\boldsymbol{\theta}}_h = 0$.

Assume that you are given $\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}^0, \Omega)$.
Can you break down $\boldsymbol{v}_h$ into local contributions $\boldsymbol{v}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}^0, \omega^{\boldsymbol{a}})$?
Unfortunately, you cannot do it (from local computations) in general!

However, we "discovered" that it is feasible if one further requires that

$$(\boldsymbol{v}_h, \boldsymbol{r}_0)_\Omega = 0 \quad \forall \boldsymbol{r}_0 \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h).$$

Step 2: the solution

Assume that $\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}^0, \Omega)$ satisfies

$$(\boldsymbol{v}_h, \boldsymbol{r}_0)_\Omega = 0 \quad \forall \boldsymbol{r}_0 \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h).$$

We can define $\boldsymbol{v}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}^0, \omega^{\boldsymbol{a}})$ on each $K \in \mathcal{T}_{\boldsymbol{a}}$ with

$$\boldsymbol{v}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{w}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_{p+1}(K) \\ \boldsymbol{\nabla} \cdot \boldsymbol{w}_h^{\boldsymbol{a}} = 0 \text{ in } K \\ \boldsymbol{w}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}_K = \psi^{\boldsymbol{a}} \boldsymbol{v}_h \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{w}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}} \boldsymbol{v}_h\|_K,$$

because the compatibility condition is satisfied:

$$(\psi^{\boldsymbol{a}} \boldsymbol{v}_h \cdot \boldsymbol{n}_K, 1)_{\partial K} = (\psi^{\boldsymbol{a}}, \boldsymbol{v}_h \cdot \boldsymbol{n}_K)_{\partial K} = (\boldsymbol{\nabla} \psi^{\boldsymbol{a}}, \boldsymbol{v}_h)_K + (\psi^{\boldsymbol{a}}, \boldsymbol{\nabla} \cdot \boldsymbol{v}_h)_K = 0$$

since $\boldsymbol{\nabla} \psi^{\boldsymbol{a}} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h)$.

This solves the problem, since after some computations, one can show that

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{v}_h^{\boldsymbol{a}} = \boldsymbol{v}_h.$$

Step 1: the natural idea

We want local contributions $\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}})$ such that

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J}.$$

The "natural candidate" is $\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$, but it does not work.

The first natural idea is then to set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J}}} \|\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}}.$$

Unfortunately, it does not lead to $(\widehat{\boldsymbol{\theta}}_h, \boldsymbol{r}_0)_\Omega = 0$ for all $\boldsymbol{r}_0 \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h)$.

Step 1: the key idea

The key idea is to over-constrain the minimization problem to ensure that

$$(\widehat{\boldsymbol{\theta}}_h, \boldsymbol{r}_0)_\Omega = 0 \qquad \boldsymbol{r}_0 \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h).$$

We then set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ (\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{r}_0)_{\omega^{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J}}} \|\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}},$$

with the additional contraint that

$$(\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{r}_0)_{\omega^{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{r}_0 \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_{\boldsymbol{a}}).$$

This would solve the problem, but is this a sound definition?

Step 1: the problem?

We want to set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega^{\boldsymbol{a}}) \\ (\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{r}_0)_{\omega^{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J}}} \|\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}},$$

but we have to check that the minimization set is not empty.

The issue is that we could take $\boldsymbol{r}_0 = \boldsymbol{\nabla} q$ for $q \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega^{\boldsymbol{a}})$, and then

$$0 = (\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{\nabla} q)_{\omega^{\boldsymbol{a}}}$$

and after integration by parts

$$(\boldsymbol{\nabla} \cdot \boldsymbol{v}_h, q) = -(\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{\nabla} q)_{\omega^{\boldsymbol{a}}}.$$

There are duplicated constraints on the divergence!

Step 1: the magic of Galerkin orthogonality strikes again

Recall that we have the “hidden” constraint that

$$(\nabla \cdot \boldsymbol{v}_h, q) = -(\nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h, \nabla q)_{\omega^{\boldsymbol{a}}} \quad \forall q \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega^{\boldsymbol{a}})$$

which could interfere with the divergence constraint $\nabla \cdot \boldsymbol{v}_h = -\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{J}$.

Fortunately, we can show that

$$\begin{aligned} -(\nabla \psi^{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h, \nabla q)_{\omega^{\boldsymbol{a}}} &= (\nabla \times \boldsymbol{A}_h, \nabla \psi^{\boldsymbol{a}} \times \nabla q)_{\omega^{\boldsymbol{a}}} \\ &= (\nabla \times \boldsymbol{A}_h, \nabla \times (\psi^{\boldsymbol{a}} \nabla q))_{\omega^{\boldsymbol{a}}} \\ &= (\boldsymbol{J}, \psi^{\boldsymbol{a}} \nabla q) = (\psi^{\boldsymbol{a}} \boldsymbol{J}, \nabla q) \end{aligned}$$

by Galerkin orthogonality as $\psi^{\boldsymbol{a}} \nabla q \in \boldsymbol{W}_h$.

We are saved: the hidden constraint agrees with divergence constraint!

Equilibration for the magnetostatic problem

Summary of the construction

Summary of the construction

Our construction now has three steps.

Step 1: compute $\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ through over-constrained minimization problems.

Step 2: decompose $\widehat{\boldsymbol{\theta}}_h$ into a divergence free decomposition $\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$

Step 3: set $\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$, and compute $\boldsymbol{H}_h^{\boldsymbol{a}}$.

Step 1

Solve the patch-wise Raviart–Thomas element problems

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ (\boldsymbol{v} - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{A}_h, \boldsymbol{r}_0)_{\omega^{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = -\boldsymbol{\nabla}\psi^{\boldsymbol{a}} \cdot \boldsymbol{J}}} \|\boldsymbol{v}_h - \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}},$$

and assemble

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}.$$

Step 2

Solve the element-wise Raviart–Thomas problems

$$\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(K) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = 0 \text{ in } K \\ \boldsymbol{v}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}_K = \psi^{\boldsymbol{a}} \boldsymbol{v}_h \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h\|_K,$$

and set

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}.$$

Step 3

Solve the patch-wise Nédélec element problems

$$\boldsymbol{H}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\theta}_h^{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega^{\boldsymbol{a}}}.$$

The Nédélec function

$$\boldsymbol{H}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{H}_h^{\boldsymbol{a}}$$

is an equilibrated flux.

Equilibration for the magnetostatic problem

Efficiency

Efficiency

Our estimator is

$$\eta := \|\boldsymbol{H}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_\Omega,$$

and we may associate with element $K \in \mathcal{T}_h$

$$\eta_K := \|\boldsymbol{H}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_K.$$

Then, it is possible to show that

$$\eta_K \leq C_{\text{eff}} \|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\widetilde{K}}},$$

with a constant C_{eff} independent of p.

Equilibration for the magnetostatic problem

Takeaways

Takeaways

The equilibration technology of the Poisson problem extends.

We can build a Nédélec flux using small uncoupled local FE problems.
The algorithm is efficient, but its implementation is harder than for Poisson.

We obtain guaranteed upper bounds.

We also have p-robust efficiency estimates, but in slightly enlarged patches.

Numerical examples

Numerical examples

Finite regularity with polynomial rhs

Settings

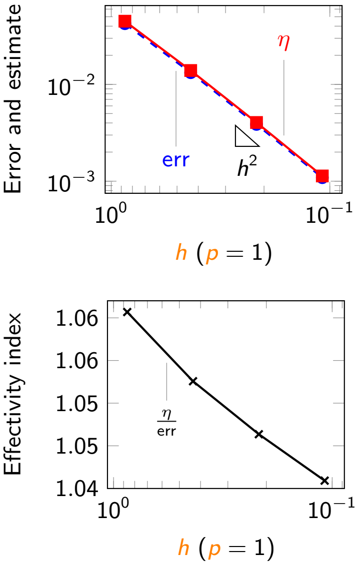
We consider the case where

$$\Omega := (0,1)^3 \qquad \boldsymbol{J} := (0,0,1).$$

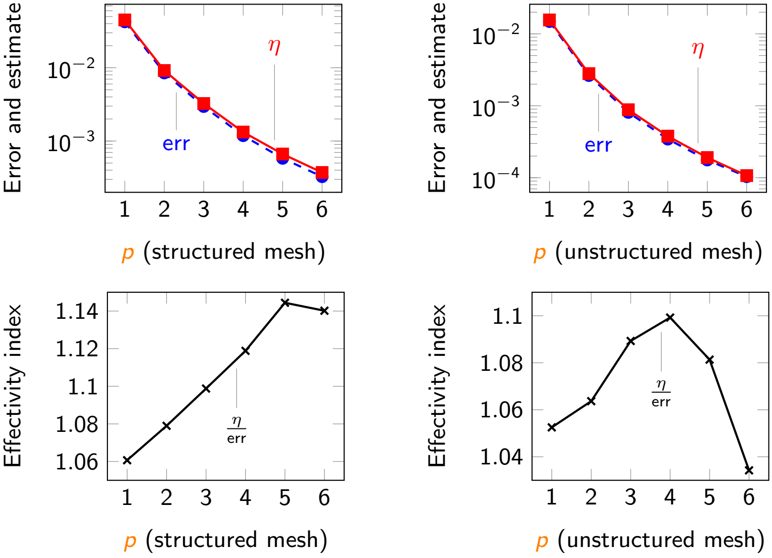
The solution is given by $\boldsymbol{A} = (0,0,A_3)$ with

$$A_3(\boldsymbol{x}) := \frac{16}{\pi^4} \sum_{n,m\geq 1} \frac{1}{nm(n^2+m^2)} \sin(n\pi \boldsymbol{x}_1)\sin(m\pi \boldsymbol{x}_2).$$

This function belongs to $H^3(\Omega)$ but not to $H^4(\Omega)$.


Error and estimate
10^{-2}
10^{-3}
10^{0}
10^{-1}
err
h^2
η
h ($p = 1$)
Error and estimate
10^{-2}
10^{-3}
10^{0}
$10^{-0.5}$
err
h^2
η
h ($p = 2$)
Effectivity index
1.06
1.06
1.05
1.05
1.04
10^{0}
10^{-1}
$\frac{\eta}{\mathrm{err}}$
h ($p = 1$)
Effectivity index
1.08
1.07
1.06
1.05
10^{0}
$10^{-0.5}$
$\frac{\eta}{\mathrm{err}}$
h ($p = 2$)

p-refinements



Numerical examples

Analytical solution with a general right-hand side

Settings

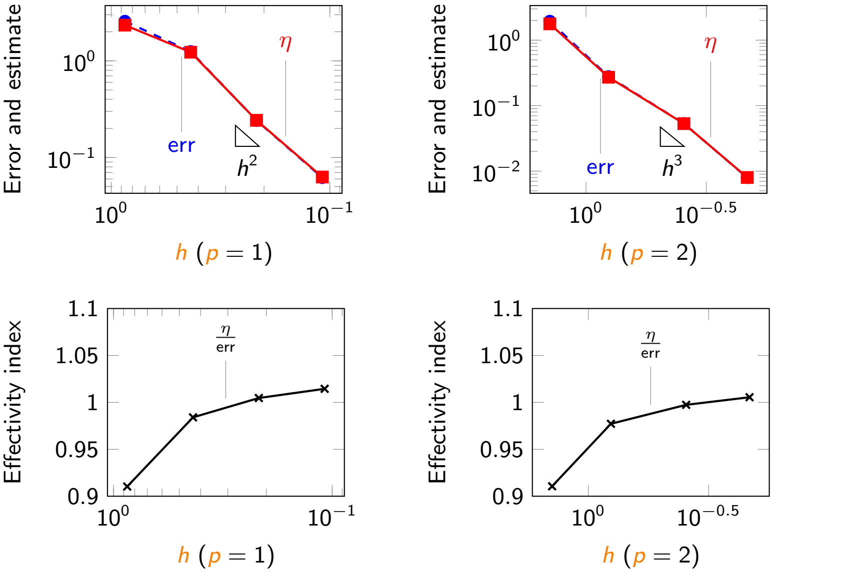
We now consider the case where

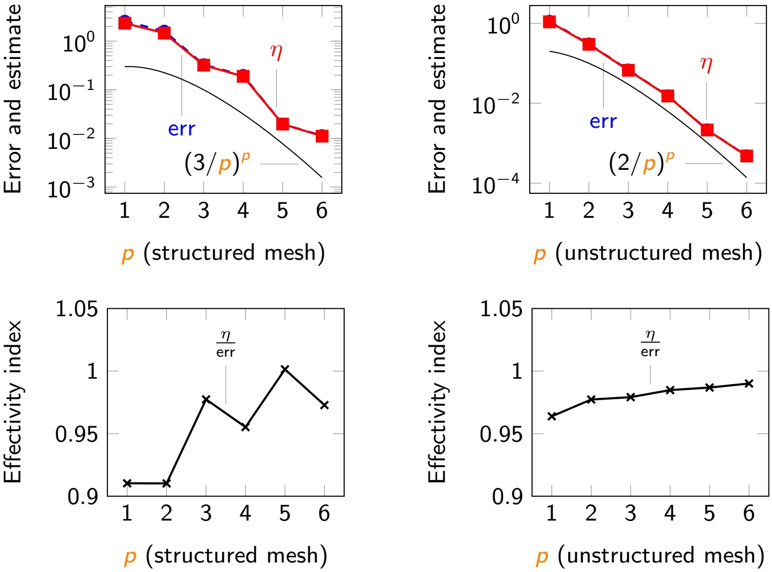
$$\Omega := (0,1)^3 \qquad \boldsymbol{J} := 8\pi^2(\sin(2\pi \boldsymbol{x}_2)\sin(2\pi \boldsymbol{x}_3), 0, 0).$$

The associated solution is analytic:

$$\boldsymbol{A} := (\sin(2\pi \boldsymbol{x}_2)\sin(2\pi \boldsymbol{x}_3), 0, 0).$$

h-refinements




Error and estimate
10^0
10^{-1}
10^{-2}
10^{-3}
η
err
$(3/p)^p$
1 2 3 4 5 6
p (structured mesh)
Error and estimate
10^0
10^{-2}
10^{-4}
η
err
$(2/p)^p$
1 2 3 4 5 6
p (unstructured mesh)
Effectivity index
1.05
1
0.95
0.9
$\frac{\eta}{\text{err}}$
1 2 3 4 5 6
p (structured mesh)
Effectivity index
1.05
1
0.95
0.9
$\frac{\eta}{\text{err}}$
1 2 3 4 5 6
p (unstructured mesh)

Numerical examples

Adaptivity with a singular solution

Settings

We consider an L-shape example where $\Omega := L \times (0,1)$, with

$$L := \{\boldsymbol{x} = (r\cos\theta, r\sin\theta);\ |\boldsymbol{x}_1|, |\boldsymbol{x}_2| \leq 1, \quad 0 \leq \theta \leq 3\pi/2\}\,.$$

The right-hand side $\boldsymbol{J}$ is non-polynomial and chosen such that

$$\boldsymbol{A}(\boldsymbol{x}) = \big(0, 0, \chi(r) r^{\alpha} \sin(\alpha\theta)\big),$$

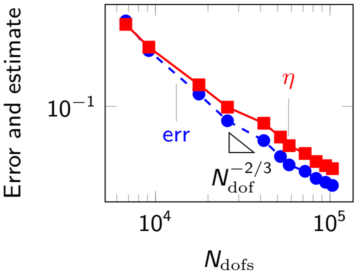
where

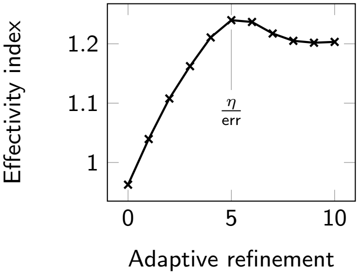
$$\alpha := 3/2, \quad r^2 := |\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2, \quad (\boldsymbol{x}_1, \boldsymbol{x}_2) = r(\cos\theta, \sin\theta),$$

and $\chi : (0,1) \to \mathbb{R}$ is a smooth cutoff function.

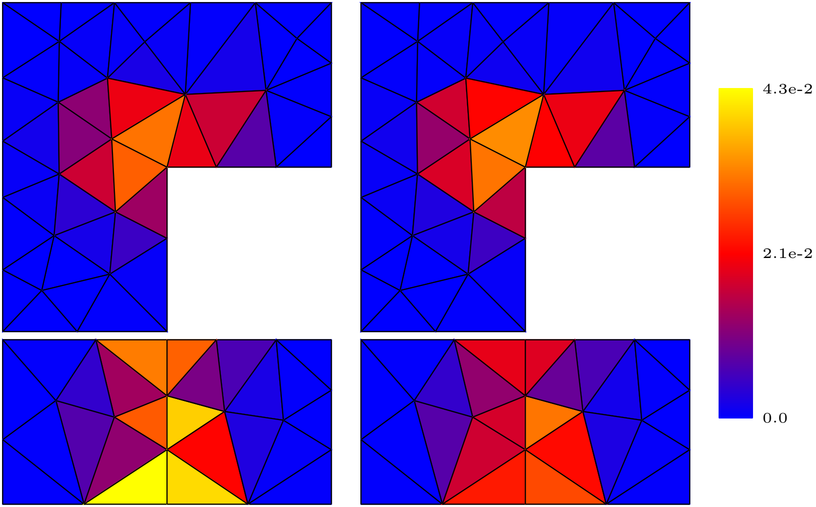
We couple the estimator with Dörfler's marking to construct adaptive meshes.
We select $p = 2$ and an initial mesh made of 415 elements.

Adpative h-refinements



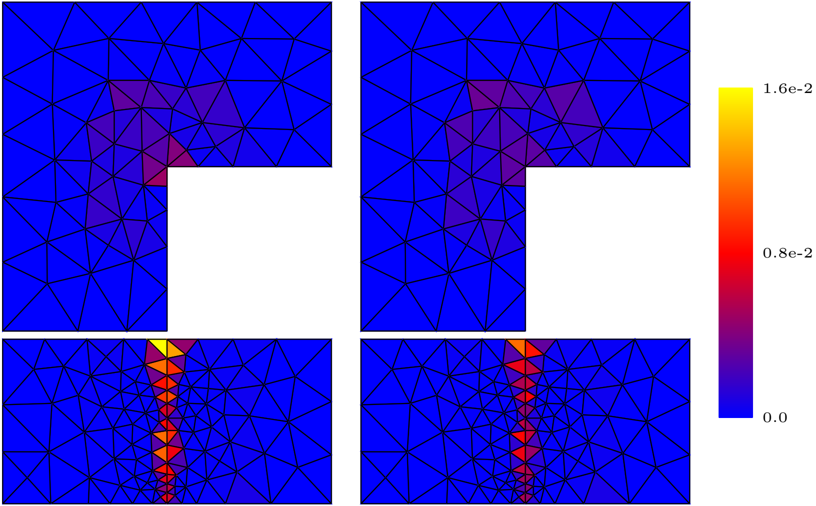


Initial mesh and estimator



Estimated error Actual error

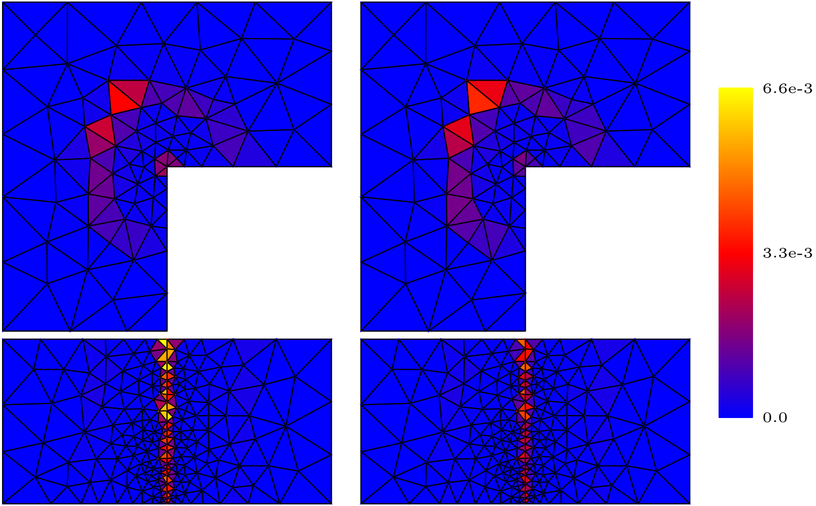
Mesh and estimator at iterator # 5



Estimated error

Actual error

Mesh and estimator at iterator # 10



Estimated error

Actual error

Concluding remarks

Takeaways

Takeaways

We now have two approaches to build equilibrated estimators in $\boldsymbol{H}(\mathbf{curl})$:

- Gedicke, Geevers, Perugia and Schöberl, 2021
- Chaumont-Frelet and Vohralík, 2021

Both approaches lead to gauranteed and p-robust error bounds.

Both algorithms are efficient, but have “tricky” implementations.

It is possible to handle more complicated problems:

- Chaumont-Frelet, soon: time-harmonic Maxwell's equations

Simplified equilibration strategies (with coarser upper bounds) are possible:

- Chaumont-Frelet, Ern and Vohralík, 2021: broken patchwise equilibration
- Chaumont-Frelet, 2021: alternative Prager–Synge