GeM - Institut de Recherche en Génie Civil et Mécanique
UMR 6183 - Nantes Université - École Centrale de Nantes - CNRS

Error estimation in reliability analysis : multi-fidelity meta-models for the estimation of probability of failure

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2022 Summer school on numerical analysis
Certification of errors in numerical simulations

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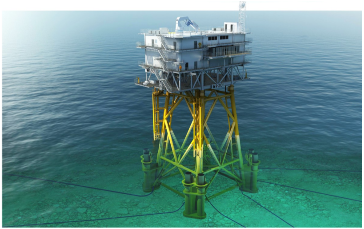
Design and uncertainties



Offshore substation in Gode Wind 3
(© Chantiers de l'Atlantique)

Floatting wind turbine

Deterministic design and certification are irrelevant for many industrial applications
⇒ need to take into account uncertainties at the design stage
⇒ need to take into account uncertainties in numerical simulations

Deterministic design

deterministic design
with safety factor

semi-probabilistic design
(Eurocodes)

Fully probabilistic design

Mechanical framework

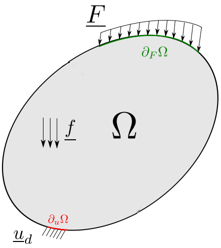


Figure – Domain Ω with imposed forces and displacements

- Quasi-static problem
- Small strain hypothesis
- Linear elastic material

Uncertainties can be on :

- The loading
- The geometry
- The material elastic properties

Kinematically admissible fields (KA)

$$\mathrm{KA}(\Omega) = \left\{ \underline{u} \in \left(\mathrm{H}^1(\Omega)\right)^d,\ \underline{u} = \underline{u}_d \text{ sur } \partial\Omega \cap \partial_u\Omega \right\}$$

Statically admissible fields (SA)

$$\mathrm{SA}(\Omega) = \left\{ \underline{\underline{\tau}} \in \left(\mathrm{L}^2(\Omega)\right)^{d\times d}_{\text{sym}};\ \forall \underline{v} \in \mathrm{KA}^0(\Omega),\ \int_\Omega \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = \int_\Omega \underline{f}\cdot\underline{v} d\Omega + \int_{\partial_F\Omega\cap\partial\omega} \underline{F}\cdot\underline{v} dS \right\}$$

Constitutive relation $\underline{\underline{\sigma}} = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u})$

Mechanical problem [Ladeveze 75]

Find $\left(\underline{u}_{ex}, \underline{\underline{\sigma}}_{ex}\right) \in \mathrm{KA}(\Omega) \times \mathrm{SA}(\Omega)$ such that
$e_{CR_\Omega}(\underline{u}_{ex}, \underline{\underline{\sigma}}_{ex}) = \|\underline{\underline{\sigma}}_{ex} - \mathbb{H} : \underline{\underline{\varepsilon}}\left(\underline{u}_{ex}\right)\|_{\mathbb{H}^{-1}}\Omega = 0$

with $\|\underline{\underline{x}}\|_{\mathbb{H},\Omega} = \sqrt{\int_\Omega \left(\underline{\underline{x}} : \mathbb{H} : \underline{\underline{x}}\right) d\Omega}$

Weak formulation

Find $\underline{u}_{ex} \in \mathrm{KA}(\Omega)$ such that
$\forall \underline{v} \in \mathrm{KA}^0(\Omega),\ a(\underline{u}_{ex}, \underline{v}) = L(\underline{v})$
with L linear form and a bilinear form

- **Subspace finite dimension of kinematically admissible fields (KA_H)**

$$\mathrm{KA}_H(\Omega_H) = \left\{ \underline{u}_H \in \left(\mathrm{H}^1(\Omega_H)\right)^d ,\ \underline{u} = \underline{u}_d \text{ on } \partial\Omega_H \cap \partial_u\Omega \right\}$$

- **Subspace of statically admissible fields (SA_H)**

$$\mathrm{SA}_H(\Omega_H) = \left\{ \underline{\underline{\tau}} \in \left(\mathrm{L}^2(\Omega_H)\right)^{d\times d}_{\mathsf{sym}} ;\ \forall \underline{v}_H \in \mathrm{KA}^0_H(\Omega_H), \right.$$
$$\left. \int_{\Omega_H} \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}\left(\underline{v}_H\right) d\Omega = \int_{\Omega_H} \underline{f} \cdot \underline{v}_H d\Omega + \int_{\partial_F\Omega\cap\partial\Omega_H} \underline{F} \cdot \underline{v}_H dS = L(\underline{v}_H) \right\}$$

- **Behaviour** $\underline{\underline{\sigma}}_H = \mathbb{H} : \underline{\underline{\varepsilon}}\left(\underline{u}_H\right)$

Finite element problem

Find $\underline{u}_H \in \mathrm{KA}(\Omega_H)$ such that
$\forall \underline{v}_H \in \mathrm{KA}^0(\Omega_H),\ \int_{\Omega_H} \underline{\underline{\sigma}}_H : \underline{\underline{\varepsilon}}\left(\underline{v}_H\right) d\Omega = \int_{\Omega_H} \underline{f} \cdot \underline{v}_H d\Omega + \int_{\partial_F\Omega\cap\partial\Omega_H} \underline{F} \cdot \underline{v}_H dS$

Performance function and probability of failure

Performance function (or limit state function)

$$G := R - S = R - \widetilde{L}(\underline{u})$$

with R the resistance and S the solicitation. $G = 0$ is the limit sate.

We define dwo domains

- $G \leq 0$: **failure domain**
- $G > 0$: **safety domain**

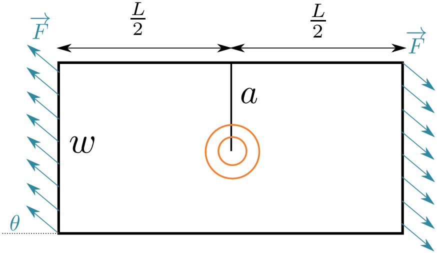
In linear mechanics, G is monotonic with respect to R and S. Both R and S can be random.
Let $\underline{X}$ be the vector gathering the random variables and p the joint distribution of random variables

Probability of failure

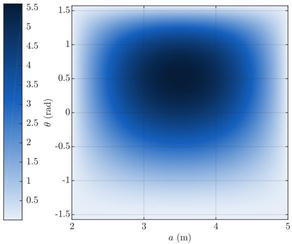
$$P_f = \int_{G(\underline{x})\leq 0} p(\underline{x})d\underline{x} = \int \mathbb{I}(G(\underline{x}) \leq 0)p(\underline{x})d\underline{x}$$

Crack opening according to Griffith's criterion :

- $R = K_{lim} = 22MPa\sqrt{m}$ is the critical stress intensity factor and $S = K_I$ is the stress intensity factor in mode I.
- $X = [a;\theta]$ is the vector containing the two random variables
$\Rightarrow G(\underline{X}) = R - S(\underline{X}) = R - S(a,\theta)$



cracked plate



joint probability density function

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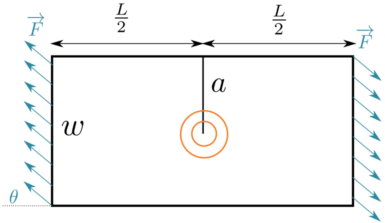

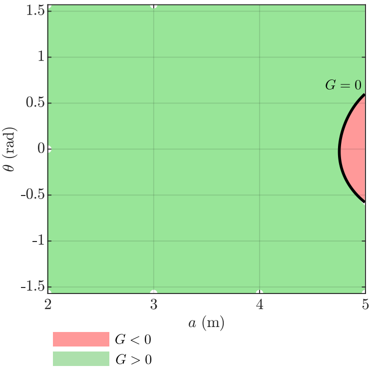

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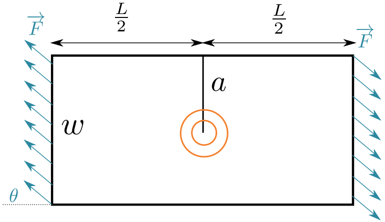

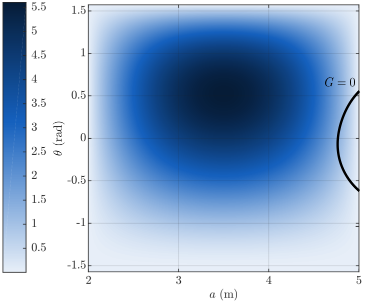

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Outline

Methods to estimate the failure probability

$$P_f = \int_{G(\underline{x})\leq 0} p(\underline{x})dx = \int \mathbb{I}(G(\underline{x}) \leq 0)p(\underline{x})d\underline{x}$$

Several techniques exist :

- Sampling : Monte Carlo estimators and variance reduction techniques subset simulations [Au 2016], MLMC [Giles 2009], ACVT [Rashki 2018], ...
- Approximating :
  - the limit state $G = 0$
    - methods FORM, SORM [Hasofer 1974]
    - support vectors machines [Vapnik 2013]
  - the performance function G by metamodels (+Monte Carlo sampling) : kriging [Krige 1951], polynnomes [Wiener 1938], neural networks [Anthony 2009], ...
  - the mechanical solution $\underline{u}$ with stochastic finite elements [Ganhem 2003]

In this presentation, I consider support vector machines to approximate $G = 0$ used with Monte Carlo sampling.
For kriging, we developped methods, see [Mell 2020].

Let assume that I have an approximation $\widehat{G}$ of G that has been **built from** m **observations** $\left(G_H(\underline{x}_i) = R - S(\underline{u}_H)\right)_{i=1..m}$**, that is to say** m **calls to the finite element code.**
We can estimated the probability of failure :

$$P_f = \int_{G(\underline{x})\leq 0} p(\underline{x})d\underline{x} \simeq \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \mathbb{I}_{\hat{G}\leq 0}(\underline{x}_i)$$

We introduced :

- The sampling error due to the finite size of the Monte Carlo population : **can be controled by making sure that** $\text{COV} = \sqrt{\frac{1-P_f}{P_f \times n_{MC}}} < \zeta$
- The approximation error due to the use of the metamodel $\widehat{G}$ instead of G : **can be reduced with adaptive learning**

Question

What is the influence of discretization error on P_f ?

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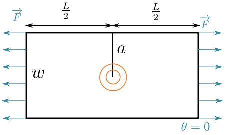


- For $\theta = 0$, G_{ex} is known $\Rightarrow$ exact computation of P_f possible
- Only one random variable a
- Kriging-based meta-model [Echard 2011] for different uniform meshes of sizes h and 2 resistances K_{lim}
- Demanding criteria on the Monte Carlo population and learning $\Rightarrow$ no influence of the estimation and approximation errors

	$K_{lim} = 9MPa\sqrt{m}$		$K_{lim} = 14MPa\sqrt{m}$	
h	P_f	err	P_f	err
0.5	$1.40\ 10^{-1}$	0.38	0	1
0.3	$1.82\ 10^{-1}$	0.20	$3.79\ 10^{-5}$	0.99
0.2	$1.94\ 10^{-1}$	0.14	$1.10\ 10^{-4}$	0.90
0.1	$2.11\ 10^{-1}$	0.07	$2.46\ 10^{-3}$	0.59
0.05	$2.19\ 10^{-1}$	0.03	$3.91\ 10^{-3}$	0.35
0.02	$2.24\ 10^{-1}$	0.01	$4.93\ 10^{-3}$	0.17
exact	$2.27\ 10^{-1}$	0	$5.98\ 10^{-3}$	0

The optimal mesh size depends on the resistance. If R is a random variable, it is impossible to choose the mesh a priori.

Outline

We introduced the subspace KA_H. We use here the **error in constitutive relation** [Ladeveze 1983]

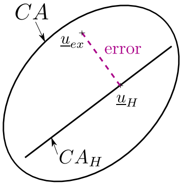


Figure – Error in constitutive relation

Pragger-Synge theorem

$$\|\underline{\underline{\varepsilon}}\left(\underline{u}_{ex}-\underline{u}_H\right)\|_{\mathbb{H}^{-1}} = |||\underline{u}_{ex}-\underline{u}_H||| \leq e_{CR_\Omega}(\hat{\underline{u}}, \hat{\underline{\underline{\sigma}}}) \ , \forall \hat{\underline{u}} \in \mathrm{KA}(\Omega) \text{ and } \forall \hat{\underline{\underline{\sigma}}} \in \mathrm{SA}(\Omega)$$

- $\hat{\underline{u}} = \underline{u}_H \in \mathrm{KA}_H$ ✓
- $\underline{\underline{\sigma}}_H = \mathbb{H} : \underline{\underline{\varepsilon}}\left(\underline{u}_H\right) \notin \mathrm{SA}(\Omega)$
  Building a SA field $\underline{\underline{\sigma}}_H$: complex but possible : EET [Ladeveze 1983], Flux-Free [Pares 2006], EESPT [Ladeveze 2012], STARFLEET [Rey V. 2014]

Error on the quantity of interest S than on G

Error on the quantity of interest :

If $\widetilde{L}$ is linear, $S_{ex} - S_H = \widetilde{L}(\underline{u}_{ex}) - \widetilde{L}(\underline{u}_H) = \widetilde{L}(\underline{u}_{ex} - \underline{u}_H)$

Adjoint problem :

Weak formulation

Find $\underline{\widetilde{u}}_{ex} \in \mathrm{KA}^0(\Omega)$ such that
$\forall \underline{v} \in \mathrm{KA}^0(\Omega),\ a(\underline{\widetilde{u}}_{ex}, \underline{v}) = \widetilde{L}(\underline{v})$

Finite element problem on the same mesh H

Find $\underline{\widetilde{u}}_H \in \mathrm{KA}^0_H(\Omega)$ such that
$\forall \underline{v} \in \mathrm{KA}^0_H(\Omega),\ a(\underline{\widetilde{u}}_H, \underline{v}) = \widetilde{L}(\underline{v})$

Bounds on S_{ex} [Becker Rannacher 1996, Ladeveze 2008]

$$S_m - \frac{1}{2} e_{CR_\Omega}(\underline{u}_H, \underline{\underline{\hat{\sigma}}}_H) e_{CR_\Omega}(\underline{\widetilde{u}}_H, \underline{\underline{\hat{\widetilde{\sigma}}}}_H) \leq S_{ex} \leq S_m + \frac{1}{2} e_{CR_\Omega}(\underline{u}_H, \underline{\underline{\hat{\sigma}}}_H) e_{CR_\Omega}(\underline{\widetilde{u}}_H, \underline{\underline{\hat{\widetilde{\sigma}}}}_H)$$

with

$$S_m = S_H - \int_\Omega \frac{1}{2}(\underline{\underline{\hat{\widetilde{\sigma}}}}_H + \mathbb{H} : \underline{\underline{\varepsilon}}\left(\underline{\widetilde{u}}_H\right)) : \mathbb{H}^{-1} : (\underline{\underline{\hat{\sigma}}}_H - \mathbb{H} : \underline{\underline{\varepsilon}}\left(\underline{u}_H\right)) d\Omega$$

We obtain **bounds on G_{ex}**

$$G^- \leq G_{ex} := R - S_{ex} \leq G^+$$

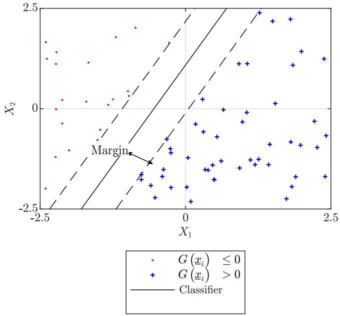
Outline

SVM with linear separator [Vapnik 2013]

Objective

To build a classifier D from n observations $(\underline{x}_i, y_i)_{i=1..n}$ where $y_i = \text{sign}(G_H(\underline{x}_i)) \in \{-1; 1\}$

- **In the case of linearly separable observations**, classifier D is built from function $f(x) = \underline{v}^T \underline{x} + a$ with $D(\underline{x}) = \text{sign}(f(\underline{x}))$.
- Parameters $\underline{v} \in \mathbb{R}^q$ and $a \in \mathbb{R}$ are sought to maximize the margin m.
- Two formulations (primal and dual) exist and can be solved with standard optimization algorithms



Dual formulation is : Find α_i for $i \in [1; n]$ such that :

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j \underline{x}_i^T \underline{x}_j - \sum_{i=1}^{n}\alpha_i \text{ is minimum and } \sum_{i=1}^{n}\alpha_i y_i = 0 \text{ and } \alpha_i \geq 0 \ \forall i = 1..n$$

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SVM with non linear separator [Vapnik 2013]

A kernel is used to replace $\underline{x}_i^T \underline{x}_j$ by a measure of the influence $\underline{x}_i$ on $\underline{x}_j$ noted $\kappa(\underline{x}_i, \underline{x}_j)$.

Here $\kappa(\underline{x}_i, \underline{x}_j) = \exp\left(\frac{||\underline{x}_i - \underline{x}_j||^2}{2\sigma^2}\right)$ where σ is an hyper-parameter determined by cross-validation

Non-linear dual formulation is :

Find α_i pour $i \in [1; n]$ such that $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j \kappa(\underline{x}_i, \underline{x}_j) - \sum_{i=1}^{n}\alpha_i$ is minimum and

$\sum_{i=1}^{n}\alpha_i y_i = 0$ and $0 \leq \alpha_i \leq C \; \forall i = 1..n$

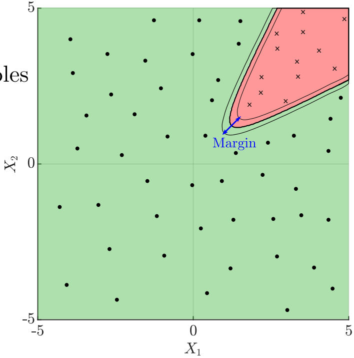
where C is the penalty parameter. Here, we choose very large $C \Rightarrow$ misclassification of observations is not allowed.

Estimation of the failure probability [Pan 2017]

Creation of the Design Of Experiment (DOE) from the joint distribution of random variables

Computation of $G_H(\underline{x}_i)$ for points i the DOE

Construction of the SVM classifier



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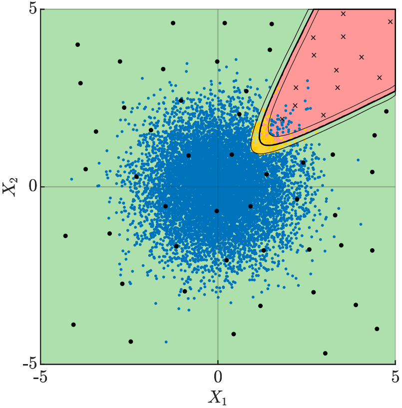
Generation of Monte Carlo population

Identification of points inside the margin

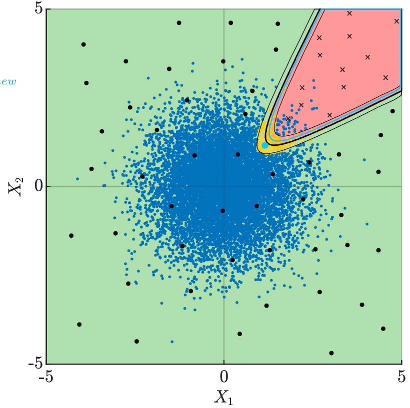


Estimation of P_f by SVM and Monte Carlo sampling [Pan 2017]

Choice of the learning point $\underline{x}_{new}$

Computation of $G_H(\underline{x}_{new})$

Update of the classifier



Once the metamodel converged, the probability of failure is estimated. If necessary, the Monte Carlo population is enlarged.

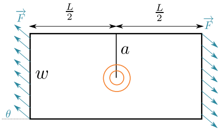
First strategy : multi-fidelity classifier

We decide to use observations $G_H(\underline{x}_i)$ to build the meta model **only if the sign of $G_H(\underline{x}_i)$ is certain, that is to say only if $G^+(\underline{x}_i)G^-(\underline{x}_i) > 0$.**

1. I define two mesh sizes h_{min} and h_{max}.
2. for every point $\underline{x}_i$ on the DOE, I compute $G_H(\underline{x}_i)$ but also $G^+(\underline{x}_i)$ and $G^-(\underline{x}_i)$ on the coarse mesh h_{max}.
3. If $G^+(\underline{x}_i)G^-(\underline{x}_i) > 0$, the point x_i is in the correct domain despite the discretization error
4. If not, I compute $G_H(\underline{x}_i)$, $G^+(\underline{x}_i)$ and $G^-(\underline{x}_i)$ on the fine mesh h_{min}
5. If $G^+(\underline{x}_i)G^-(\underline{x}_i) > 0$, the x_i is in the correct domain
6. Otherwise I keep the observation $\text{sign}(R - S_m(\underline{x}_i))$

The rest of the method is unchanged (selection of the learning point, controlling the size of the Monte Carlo population)

Once the algorithm has converged, it is possible to obtain at low cost two classifiers from $\text{sign}(G^+(\underline{x}_i))$ and $\text{sign}(G^-(\underline{x}_i))$ and to obtain two indicators P_+ and P_-



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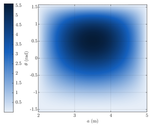
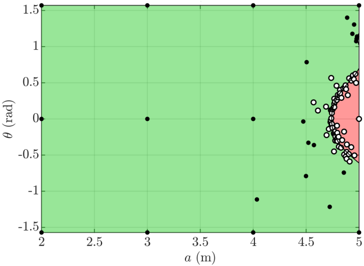
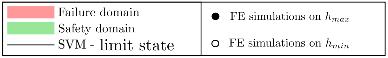

joint probability density function

Figure – P_{ref}=5.89 10^{-3}, 16 ks, h=0.02

- 2 mesh sizes $h_{max} = 0.5$ et $h_{min} = 0.1$
- 5 Monte Carlo populations
- Parameters : $\eta_1 = 10^{-4}$ and $\eta_2 = 0.02$, $n_{DOE} = 12$.



P_f ($\times 10^{-3}$)	Nb calls h_{max}	Nb calls h_{min}	P_- ($\times 10^{-3}$)	P_+ ($\times 10^{-3}$)	$t_{EF}(s)$	t_{err} (s)	t_{algo} (s)
6.91	94	69	2.92	15.31	286	9610	16.3
7.11	68	47	3.71	12.46	202	6880	6.7
7.08	73	47	3.02	14.31	172	6352	4.7
6.50	70	48	3.07	12.89	219	6976	10.6
6.80	74	49	3.18	11.80	203	6815	15



The fine mesh is used only close to the limit state. We obtain an non guaranteed error estimtion on $P_{f,ex}$ thanks to P_+ and P_-

Second strategy : double classifier [Mell 2022, submitted]

I determine a unique mesh size. I build two classifiers in parallel.

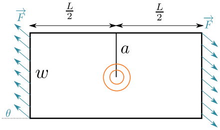
- one separating the **certainly fail population** ($G^+(\underline{x}_i) < 0$) from the rest
- one separating the **certainly safe population** ($G^-(\underline{x}_i) > 0$) from the rest

For each classifier the next learning point is defined. The FE code is called for these two points. Bounds computed are used as observations for both classifiers.

It allows to :

- compute upper bound P_+ and lower bound P_- of P_f for a given mesh
- exhibit the uncertain population

At the end, if $[P_-; P_+]$ is too large, I refine the mesh and only class the uncertain population



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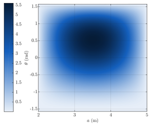
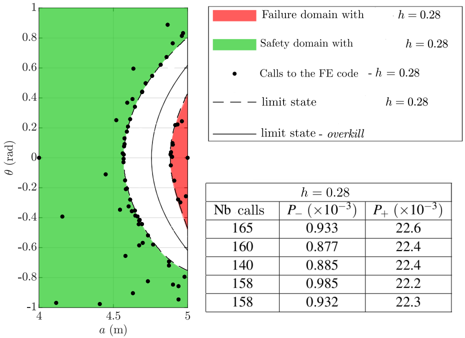


joint probability density function

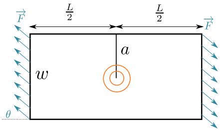
- 2 meshes $h_1 = 0.28$ then $h_2 = 0.14$
- 5 Monte Carlo populations
- Parameters : $\eta_1 = 10^{-4}$ et $\eta_2 = 0.02$, $n_{DOE} = 12$.

Figure – P_{ref}=5.89 10^{-3}



$h = 0.28$		
t_{EF} (s)	t_{err} (s)	t_{algo} (s)
101	1909	32
113	2259	31
150	2338	1231
113	1825	1040
96	1974	43

$h = 0.28$		
Nb calls	P_- ($\times 10^{-3}$)	P_+ ($\times 10^{-3}$)
165	0.933	22.6
160	0.877	22.4
140	0.885	22.4
158	0.985	22.2
158	0.932	22.3



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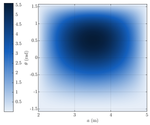

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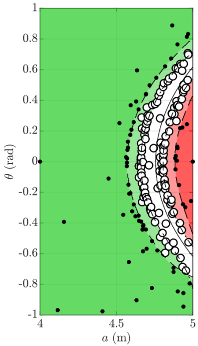
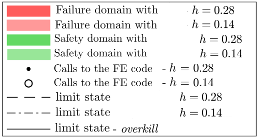
Figure – P_{ref}=5.89 10^{-3}





$h = 0.28$			$h = 0.14$		
$t_{EF}(s)$	t_{err} (s)	t_{algo} (s)	$t_{EF}(s)$	t_{err} (s)	t_{algo} (s)
101	1909	32	165	6194	24
113	2259	31	137	6844	25
150	2338	1231	172	7155	20
113	1825	1040	104	4152	17
96	1974	43	109	4329	27

$h = 0.28$			$h = 0.14$		
Nb calls	P_- ($\times 10^{-3}$)	P_+ ($\times 10^{-3}$)	Nb calls	P_- ($\times 10^{-3}$)	P_+ ($\times 10^{-3}$)
165	0.933	22.6	127	3.33	12.9
160	0.877	22.4	120	3.33	12.9
140	0.885	22.4	112	3.33	13.2
158	0.985	22.2	100	3.33	13.1
158	0.932	22.3	101	3.32	13.1

- It is crucial to control discretization error estimation in reliability analysis
- A posteriori error estimators can guide the construction of multi-fidelity meta-models : numerical effort is focused close to the limit state
- Bounds on G_{ex} can be used as observations to build meta-models to estimate bounds on the probability of failure

Prospects and open questions

- Cutting-edge adaptive remeshing techniques would enable the reduction of computational cost (generation of optimal mesh).
- Balance with approximation error : generating a large Monte Carlo population may not be necessary if the mesh size is not small enough to obtain good accuracy close to the limit state
- Time-dependant reliability raises challenges. A posteriori error estimators do exist for non-linear constitutive laws. To which extent could they be applied ?
- Towards reliability-based design and reliability-oriented computations
- Balance with modeling error (mechanical model, choice of distribution of the random variables, ...)

Thank you

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