

On quasi-optimal computational costs of adaptive FEM

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FWF



What is all about?

A nonlinear model problem

AFEM with exact solver

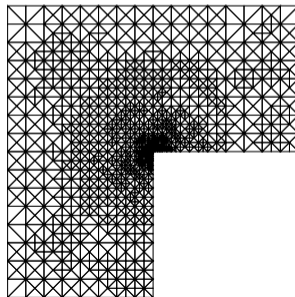
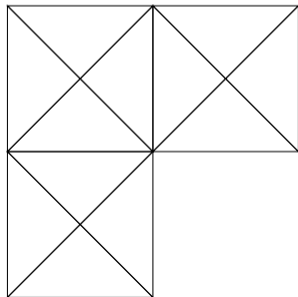
AFEM with iterative solver

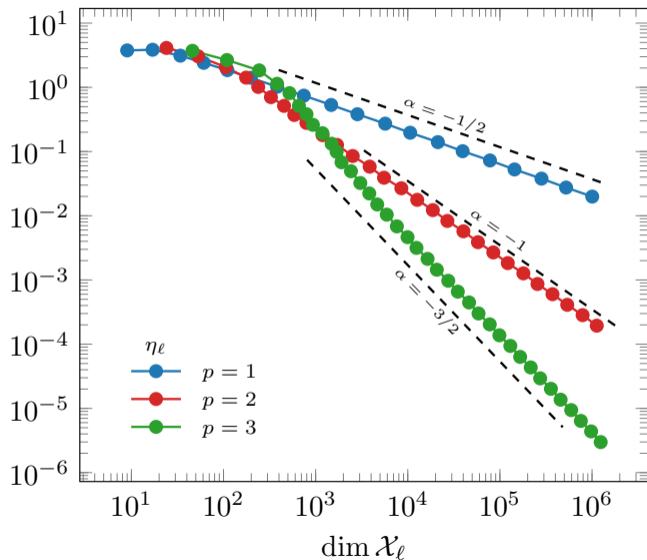
Numerical experiment

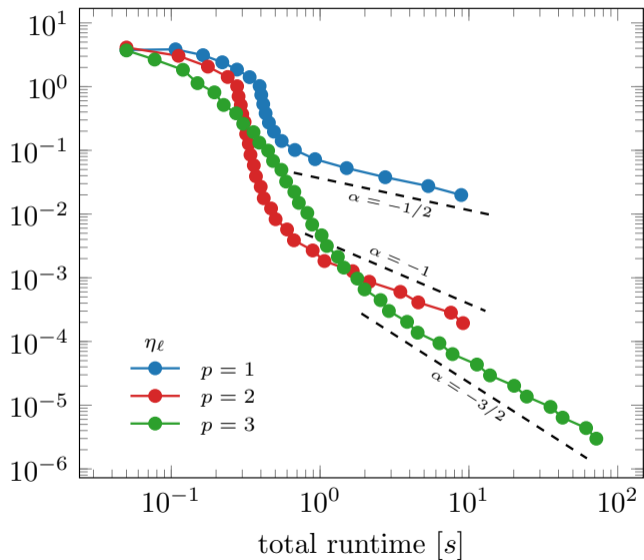
Conclusion & three notable extensions

What is all about?

- solve $-\Delta u = 0$ in Ω subject to appropriate boundary conditions
- prescribed exact solution $u(x) = r^{2/3} \sin(2\varphi/3)$
- adaptivity driven by residual error estimator







A nonlinear model problem

Model problem

$$\begin{aligned} -\operatorname{div} \mathcal{A}(\nabla u^*) &= 0 && \text{in } \Omega \\ u^* &= 0 && \text{on } \partial\Omega \end{aligned}$$

Example for strongly monotone nonlinearities

- nonlinear material laws $\mathbf{M} = \chi(|\mathbf{H}|) \mathbf{H}$ in magnetostatics
- together with $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$
- e.g., $\mathcal{A}(\nabla u) = \left(1 + \frac{1}{\sqrt{1 + |\nabla u|^2}}\right) \nabla u$


- \mathcal{H} separable Hilbert space with norm $\|\cdot\|$, \mathcal{X}_ℓ closed subspace
- nonlinear operator $\mathbf{A}: \mathcal{H} \rightarrow \mathcal{H}^*$ such that, for all $u, v \in \mathcal{H}$,
 - (O1) strongly monotone $\alpha \|u - v\|^2 \leq \langle \mathbf{A}u - \mathbf{A}v, u - v \rangle$
 - (O2) Lipschitz continuous $\|\mathbf{A}u - \mathbf{A}v\| \leq L \|u - v\|$

Main Theorem on Strongly Monotone Operators (Zarantonello '60)

- exists unique $u^* \in \mathcal{H}$ such that $\langle \mathbf{A}u^*, v \rangle = 0$ for all $v \in \mathcal{H}$

Corollary

- exists unique $u_\ell^* \in \mathcal{X}_\ell$ such that $\langle \mathbf{A}u_\ell^*, v_\ell \rangle = 0$ for all $v_\ell \in \mathcal{X}_\ell$
- $\|u^* - u_\ell^*\| \leq \frac{L}{\alpha} \min_{u_\ell \in \mathcal{X}_\ell} \|u^* - u_\ell\|$

 Zarantonello: U.S. Army Math. Res. Center, Tech. Report, 160 (1960)

 Zeidler: Nonlinear functional analysis and its applications, Part II/B

- **constructive proof** by Banach fixpoint theorem

- $I_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}^*$ Riesz mapping $\langle I_{\mathcal{H}}u, v \rangle = (u, v)_{\mathcal{H}}$


- $\Phi(u) := u - \delta I_{\mathcal{H}}^{-1} \mathbf{A}u$ with $0 < \delta < 2\alpha/L^2$

$$\implies \|\Phi(u) - \Phi(v)\| \leq \kappa \|u - v\| \quad \text{with} \quad \kappa := [1 - \delta(2\alpha - \delta L^2)]^{1/2}$$

1 Φ has unique fixpoint $u^* \in \mathcal{H}$ ($\iff \mathbf{A}u^* = 0$)

2 $u^0 \in \mathcal{H}$ arbitrary, $u^k := \Phi(u^{k-1})$

$$\implies \|u^* - u^k\| \leq \frac{\kappa}{1 - \kappa} \|u^k - u^{k-1}\| \leq \frac{\kappa^k}{1 - \kappa} \|u^1 - u^0\|$$

 Zarantonello: U.S. Army Math. Res. Center, Tech. Report, 160 (1960)

 Zeidler: Nonlinear functional analysis and its applications, Part II/B

- $u_\ell^0 \in \mathcal{X}_\ell$ arbitrary, $u_\ell^k := \Phi_\ell(u_\ell^{k-1})$

$$\implies \|u_\ell^* - u_\ell^k\| \leq \kappa \|u_\ell^* - u_\ell^{k-1}\| \quad \text{for all } k \in \mathbb{N}$$

- triangle inequality $\implies \frac{1 - \kappa}{\kappa} \|u_\ell^* - u_\ell^k\| \leq \|u_\ell^k - u_\ell^{k-1}\| \leq (1 + \kappa) \|u_\ell^* - u_\ell^{k-1}\|$

Discrete Zarantonello iteration

- solve $(u_\ell^k, v_\ell)_\mathcal{H} = (u_\ell^{k-1}, v_\ell)_\mathcal{H} + \frac{\alpha}{L^2} \langle \mathbf{A}u_\ell^{k-1}, v_\ell \rangle$ for all $v_\ell \in \mathcal{X}_\ell$
- i.e., each step of Zarantonello iteration solves one Laplace problem



Congreve, Wihler: J. Comput. Appl. Math., 311 (2017)

- $k = 1, \dots, K(\ell)$ steps of Zarantonello iteration $u_\ell^k \approx u_\ell^*$ per mesh \mathcal{T}_ℓ
- index set $(\ell, k) \in \mathcal{Q} \subset \mathbb{N}_0^2$ of discrete approximations u_ℓ^k
- suppose cost $\mathcal{O}(\#\mathcal{T}_\ell)$ for linear solve / estimate / mark / refine on mesh \mathcal{T}_ℓ

Aim for thorough proof of

$$\|u^*\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \left(N^s \min_{\#\mathcal{T}_{\text{opt}} \leq N} [\|u^* - u_{\text{opt}}^* \| + \eta_{\text{opt}}(u_{\text{opt}}^*)] \right) \quad \text{for all } s > 0$$

$$\stackrel{\checkmark}{\approx} \sup_{(\ell, 0) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s [\|u^* - u_\ell^* \| + \eta_\ell(u_\ell^*)]$$

$$\stackrel{?}{\approx} \sup_{(\ell, k) \in \mathcal{Q}} \left(\sum_{\substack{(\ell', k') \in \mathcal{Q} \\ (\ell', k') \leq (\ell, k)}} \#\mathcal{T}_{\ell'} \right)^s [\|u^* - u_\ell^k \| + \eta_\ell(u_\ell^k)]$$

AFEM with exact solver

Input: \mathcal{T}_0 , $0 < \theta \leq 1$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE:** compute u_ℓ^*
- **ESTIMATE:** compute $\eta_\ell(T, u_\ell^*)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^*)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^*)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete solutions u_ℓ^* and corresponding estimators $\eta_\ell(u_\ell^*)$

- **in practice:** \mathcal{M}_ℓ is always chosen with quasi-minimal cardinality



Dörfler: SIAM J. Numer. Anal., 33 (1996)



Stevenson: Found. Comput. Math., 7 (2007)

- suppose that \mathbf{A} has a potential, i.e.,

$$(O3) \quad \mathcal{J} : \mathcal{H} \rightarrow \mathbb{R} \quad \text{such that} \quad \langle \mathbf{A}u, v \rangle = \lim_{r \rightarrow 0} \frac{\mathcal{J}(u + rv) - \mathcal{J}(u)}{r} \quad \text{for all } u, v \in \mathcal{H}$$

Lemma

- (O1) & (O2) & (O3)

- $v_\ell \in \mathcal{X}_\ell$

$$\Rightarrow \quad \frac{\alpha}{2} \|u_\ell^* - v_\ell\|^2 \leq \mathcal{J}(v_\ell) - \mathcal{J}(u_\ell^*) \leq \frac{L}{2} \|u_\ell^* - v_\ell\|^2$$

- i.e., Galerkin formulation is equivalent to minimization of \mathcal{J} over \mathcal{X}_ℓ

$$\Rightarrow \quad \|u_{\ell+1}^* - u_\ell^*\|^2 \simeq \mathcal{J}(u_\ell^*) - \mathcal{J}(u_{\ell+1}^*) = [\mathcal{J}(u_\ell^*) - \mathcal{J}(u^*)] - [\mathcal{J}(u_{\ell+1}^*) - \mathcal{J}(u^*)]$$




Theorem (Diening, Kreuzer '08; ... ; Carstensen, Feischl, Page, P. '14)

- (O1) & (O2) & (O3) & residual error estimator
- arbitrary $0 < \theta \leq 1$

\implies exists $0 < q < 1$ such that

$$\|u^* - u_\ell^*\| \lesssim \eta_\ell(u_\ell^*) \lesssim q^{\ell-\ell'} \eta_{\ell'}(u_{\ell'}^*) \quad \text{for all } \ell' \leq \ell$$

- **required:** certain Pythagoras-type quasi-orthogonality
- **here:** energy

-
-  Diening, Kreuzer: SIAM J. Numer. Anal., 46 (2008)
 -  Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)
 -  Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

Theorem (Stevenson '07; ... ; Carstensen, Feischl, Page, P. '14)

- (O1) & (O2) & (O3) & residual error estimator
 - arbitrary $s > 0$
 - $\|u^*\|_{A_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\text{opt}} \leq N} N^s \eta_{\text{opt}}(u_{\text{opt}}^*) < \infty$
 - sufficiently small $0 < \theta \leq 1$
- $\implies \sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^*) \simeq \|u^*\|_{A_s}$

- **note:** $\eta_{\text{opt}}(u_{\text{opt}}^*) \simeq [\|u^* - u_{\text{opt}}^*\| + \eta_{\text{opt}}(u_{\text{opt}}^*)]$
- **required:** linear convergence of $\eta_\ell(u_\ell^*)$

- 📄 Stevenson: Found. Comput. Math., 7 (2007)
- 📄 Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)
- 📄 Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

AFEM with iterative solver

- reliability & stability of residual error estimator

$$\begin{aligned} \|u^* - u_\ell^k\| &\leq \|u^* - u_\ell^*\| + \|u_\ell^* - u_\ell^k\| \\ &\stackrel{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^*) + \|u_\ell^* - u_\ell^k\| \\ &\stackrel{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^* - u_\ell^k\| \\ &\stackrel{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\| \end{aligned}$$

⇒ contraction allows for a-posteriori error control

- idea:** equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

⇒ stop linearization for $K = k$ with $\|u_\ell^K - u_\ell^{K-1}\| \leq \lambda \eta_\ell(u_\ell^K)$

- nested iteration:** $u_\ell^0 := u_{\ell-1}^K$

⇒ a-posteriori error control for all u_ℓ^k but u_ℓ^0

Input: \mathcal{T}_0 , u_0^0 , $0 < \theta \leq 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \dots, K$, **repeat**

- ▶ compute u_ℓ^k

- ▶ compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

until $\|u_\ell^K - u_\ell^{K-1}\| \leq \lambda \eta_\ell(u_\ell^K)$

- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

- **REFINE:** $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$

Output: Discrete solutions u_ℓ^k and corresponding estimator $\eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$

- **note:** number of solver steps $K = K(\ell)$ might vary with ℓ

Theorem (Gantner, Haberl, P., Schimanko '21)

- (O1) & (O2) & (O3) & residual error estimator
 - $|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} : u_{\ell'}^{k'} \text{ computed earlier than } u_{\ell}^k\}$
 - quasi-error $\Delta_{\ell}^k := [\|u^* - u_{\ell}^k\| + \eta_{\ell}(u_{\ell}^k)]$
 - arbitrary u_0^0 , arbitrary $0 < \theta \leq 1$, sufficiently small $\lambda > 0$
- \implies exists $0 < q < 1$ such that $\|u^* - u_{\ell}^k\| \lesssim \Delta_{\ell}^k \lesssim q^{|\ell, k| - |\ell', k'|} \Delta_{\ell'}^{k'}$

- note [CFPP14]: linear convergence $\iff \sum_{\substack{(\ell, k) \in \mathcal{Q} \\ |\ell, k| > |\ell', k'|}} \Delta_{\ell}^k \lesssim \Delta_{\ell'}^{k'}$

- for exact solvers: $\Delta_{\ell}^* \simeq \eta_{\ell}(u_{\ell}^*)$

 Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

 Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

- $\eta_\ell(u_\ell^K)$ ^{stability} $\leq \eta_\ell(u_\ell^*) + C_{\text{stab}} \|u_\ell^* - u_\ell^K\|$ ^{solver} $\leq \eta_\ell(u_\ell^*) + C_{\text{stab}} \frac{\kappa}{1 - \kappa} \|u_\ell^K - u_\ell^{K-1}\|$
 $\leq \eta_\ell(u_\ell^*) + C_{\text{stab}} \frac{\kappa}{1 - \kappa} \lambda \eta_\ell(u_\ell^K)$ ^{terminate}
- similarly:** $\eta_\ell(u_\ell^*) \leq \eta_\ell(u_\ell^K) + C_{\text{stab}} \frac{\kappa}{1 - \kappa} \lambda \eta_\ell(u_\ell^K)$

$$\implies (1 - C\lambda) \eta_\ell(u_\ell^K) \leq \eta_\ell(u_\ell^*) \leq (1 + C\lambda) \eta_\ell(u_\ell^K)$$

- moreover:** Dörfler marking for (θ, u_ℓ^K) \iff Dörfler marking for $(\tilde{\theta}, u_\ell^*)$

$$\implies \text{linear convergence of } \eta_\ell(u_\ell^K) \simeq \eta_\ell(u_\ell^*)$$

$$\implies \text{summability of } \Delta_\ell^k = \|u^* - u_\ell^k\| + \eta_\ell(u_\ell^k) \simeq \|u_\ell^* - u_\ell^k\| + \eta_\ell(u_\ell^k) \text{ by geometric series}$$

Theorem (Gantner, Haberl, P., Schimanko '21)

- arbitrary $s > 0$
- $\|u^*\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\text{opt}} \leq N} N^s \Delta_{\text{opt}}^* < \infty$
- sufficiently small $0 < \theta \leq 1$ and $\lambda > 0$

$$\Rightarrow \|u^*\|_{\mathbb{A}_s} \lesssim \sup_{(\ell, k) \in \mathcal{Q}} \left(\sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} \#\mathcal{T}_{\ell'} \right)^s \Delta_{\ell}^k \lesssim \max \{ \|u^*\|_{\mathbb{A}_s}, \Delta_0^0 \}$$

- full linear convergence

$$\Rightarrow \sup_{(\ell,k) \in \mathcal{Q}} \left(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \leq |\ell,k|}} \#\mathcal{T}_{\ell'} \right)^s \Delta_{\ell}^k \simeq \sup_{(\ell,k) \in \mathcal{Q}} (\#\mathcal{T}_{\ell})^s \Delta_{\ell}^k$$

and $\sup_{(0,k) \in \mathcal{Q}} (\#\mathcal{T}_0)^s \Delta_0^k \lesssim \Delta_0^0$

- sufficiently small $\lambda > 0$

$$\Rightarrow \sup_{\substack{(\ell,k) \in \mathcal{Q} \\ \ell > 0}} (\#\mathcal{T}_{\ell})^s \Delta_{\ell}^k \lesssim \sup_{(\ell,0) \in \mathcal{Q}} (\#\mathcal{T}_{\ell})^s \Delta_{\ell}^K \simeq \sup_{(\ell,0) \in \mathcal{Q}} (\#\mathcal{T}_{\ell})^s \Delta_{\ell}^* \simeq \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_{\ell})^s \eta_{\ell}(u_{\ell}^*)$$

- sufficiently small $0 < \theta \leq 1$

$$\Rightarrow \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_{\ell})^s \eta_{\ell}(u_{\ell}^*) \simeq \|u^*\|_{\mathbb{A}_s}$$



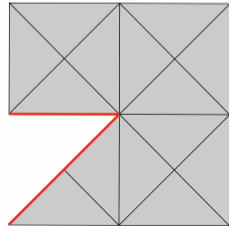
Numerical experiment

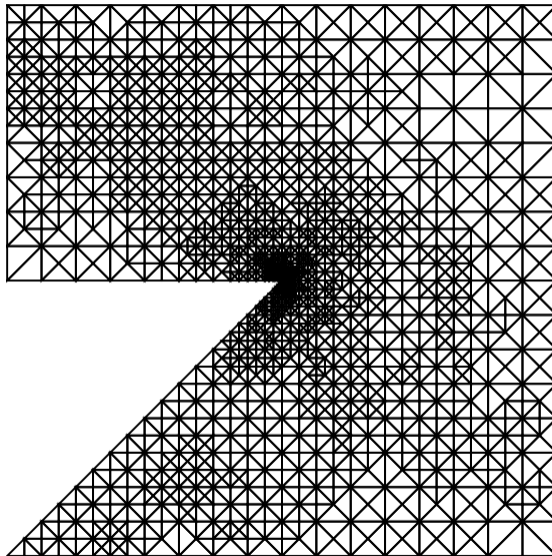
mixed BVP

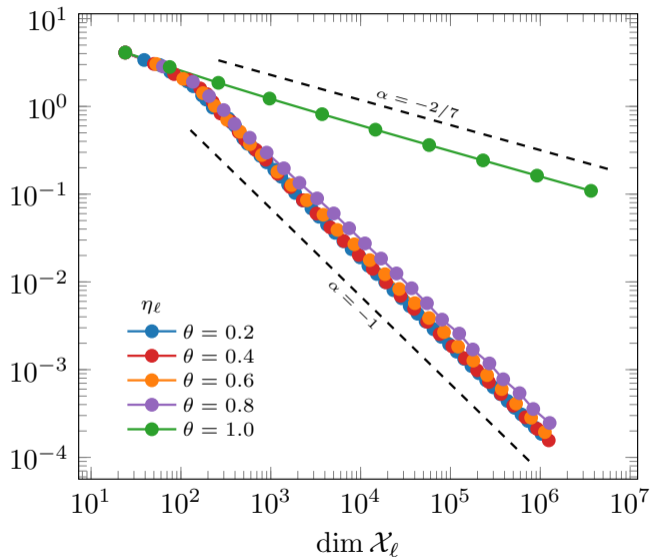
$$\begin{aligned} -\operatorname{div}[\mu(|\nabla u^\star|) \nabla u^\star] &= f && \text{in } \Omega \\ \mu(|\nabla u^\star|) \nabla u^\star \cdot \mathbf{n} &= g && \text{on } \Gamma_N \\ u^\star &= 0 && \text{on } \Gamma_D \end{aligned}$$

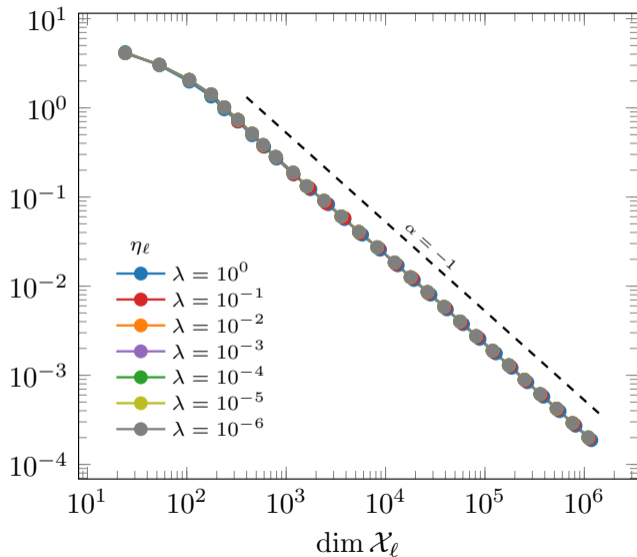
- $\mu(t) := 2 + \frac{1}{\sqrt{1+t^2}} \implies \alpha = 2, \quad L = 3 \quad \text{w.r.t.} \quad \|\cdot\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$

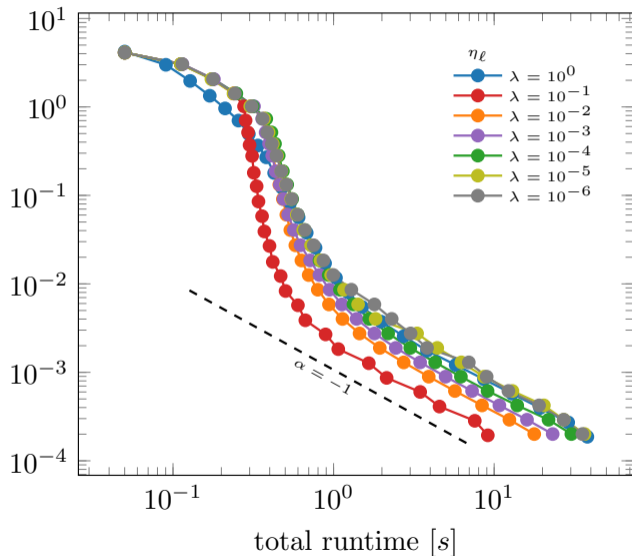
- $$\begin{aligned} \eta_\ell(T, u_\ell) &= h_T^2 \|f + \operatorname{div}[\mu(|\nabla u_\ell|) \nabla u_\ell]\|_{L^2(T)}^2 \\ &\quad + h_T \|g - \mu(|\nabla u_\ell|) \nabla u_\ell \cdot \mathbf{n}\|_{L^2(\partial T \cap \Gamma_N)}^2 \\ &\quad + h_T \|[\mu(|\nabla u_\ell|) \nabla u_\ell \cdot \mathbf{n}]\|_{L^2(\partial T \cap \Omega)}^2 \end{aligned}$$

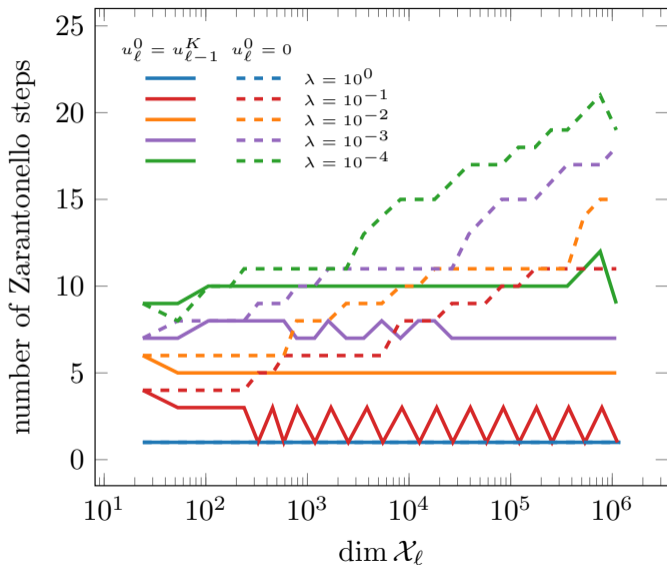


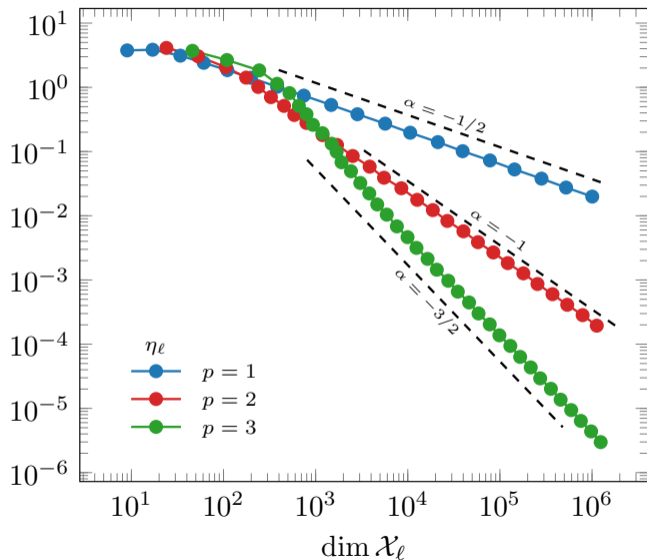


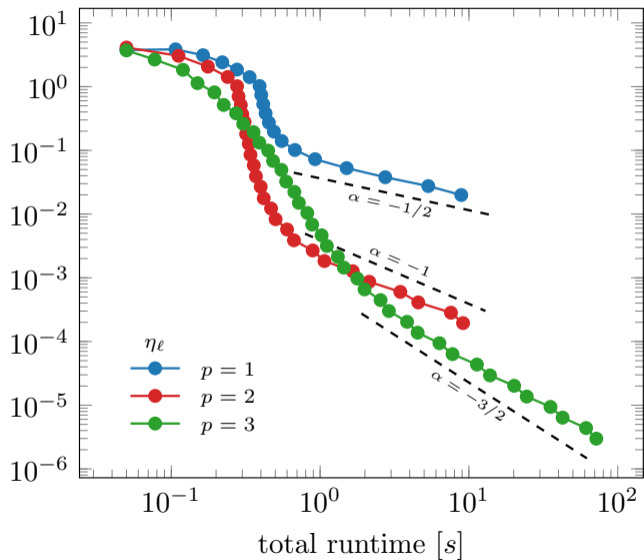












Conclusion & three notable extensions


- **new:** abstract framework for full linear convergence
 - ▶ applies to (strongly monotone) energy minimization problems
 - ▶ exploits only usual properties of residual error estimators
 - ▶ relies only on contractive linearization / solver
- **implication:** adaptive algorithm with quasi-optimal cost (instead of only optimal rate)
 - ▶ if rate-optimal, then even cost-optimal


- stop linearization if $|\mathcal{J}(u_\ell^K) - \mathcal{J}(u_\ell^{K-1})|^{1/2} \leq \lambda \eta_\ell(u_\ell^K)$
 - ▶ instead of $\|u_\ell^K - u_\ell^{K-1}\| \leq \lambda \eta_\ell(u_\ell^K)$

⇒ [HPW21] full linear convergence holds for arbitrary $\lambda > 0$
and holds for Zarantonello iteration & Kacanov iteration & damped Newton method

- **consequence:** rates with respect to costs and dofs always coincide!
- applies to linear PDEs with contractive PCG [CNX12] or multigrid [WZ17] solvers

 Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)

 Chen, Nochetto, Xu: Numer. Math., 120 (2012)

 Wu, Zheng: Appl. Numer. Math., 113 (2017)

- employ contractive algebraic solver for the linear systems, i.e., $u^* \approx u_\ell^* \approx u_\ell^k \approx u_\ell^{k,j}$
- \Rightarrow index set $(\ell, k, j) \in \mathcal{Q} \subset \mathbb{N}_0^3$

- stop algebraic solver if $\|u_\ell^{k,J} - u_\ell^{k,J-1}\| \leq \mu [\eta_\ell(u_\ell^{k,J}) + \|u_\ell^{k,J} - u_\ell^{k-1,J}\|]$
 - stop Zarantonello linearization if $\|u_\ell^{K,J} - u_\ell^{K-1,J}\| \leq \lambda \eta_\ell(u_\ell^{K,J})$
- \Rightarrow full linear convergence for sufficiently small λ and μ [HPSV21]

- full linear convergence for arbitrary λ and sufficiently small μ [MPV22+]
and extension to Zarantonello iteration & Kacanov iteration & damped Newton method

 Haberl, Praetorius, Schimanko, Vohralik: Numer. Math., 147 (2021)

 Miraci, Praetorius, Vohralik: in progress (2022+)

Model problem

$$\begin{aligned} -\operatorname{div}(A\nabla u^\star) + b(u^\star) &= f && \text{in } \Omega \\ u^\star &= 0 && \text{on } \Gamma = \partial\Omega \end{aligned}$$

- **monotonicity:** A SPD and $b(\cdot)$ smooth with $b'(\xi) \geq 0$
- **polynomial growth:** $b^{(k)}(\xi) \leq C(1 + |\xi|^{n-k})$ for some $n \in \mathbb{N}$ and all $0 \leq k \leq n$

⇒ full linear convergence for sufficiently small λ

- **difficulty:** resulting operator \mathbf{A} is strongly monotone, **but** only locally Lipschitz





Becker, Brunner, Innerberger, Melenk, Praetorius: Comput. Math. Appl., 118 (2022)



Becker, Brunner, Innerberger, Praetorius: in progress (2022+)

Thank you for your attention!

 Gregor Gantner, Alexander Haberl, Dirk Praetorius, Stefan Schimanko:
Rate optimality of adaptive finite element methods with respect to the overall computational costs
Math. Comp., 90 (2021), 2011–2040

 Michael Innerberger, Dirk Praetorius:
MooAFEM: An object oriented Matlab code for higher-order adaptive FEM for (nonlinear) elliptic PDEs
arXiv:2203.01845 **or:** homepage → software → mooafem

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NumPDEs
Workgroup on Numerics of PDEs