$hp$—AMR applied to the neutron transport equation discretized by a Discontinuous Galerkin upwind scheme
Context

- **Nuclear reactor core physics** (see F. Madiot’s presentation)
- For “Sodium Fast Reactors” (SFR), the diffusion (or SP$_N$) approximation is deemed insufficient for *core design calculations* (homogenized assemblies)
- In the early stage (2009-2012) of the *ASTRID prototype* (cancelled in 2019) design studies, development of a *3D neutron transport solver* to supplement the existing ERANOS code

**SNATCH solver**

Along the following lines:

- angular discretization: so-called *$S_N$ method* (quadrature-based):
  \[ \tilde{\Omega} \in S_2 \rightarrow (w_n, \tilde{\Omega}_n)_{1 \leq n \leq N} \]

- high-order spatial scheme → *Discontinuous Galerkin (DG) upwind scheme*

- both direct and adjoint solutions → *perturbation theory toolbox*

With the additional idea of having a modular environment to test numerical methods

- In this framework, *AMR has been investigated*

This presentation

the salient features of this (old) work + “links” to more recent works by other researchers on this subject
Contents

► **Equations system and spatial discretization**
  - Semi-discrete equations system
  - Spatial discretization

► **hp-Adaptive Mesh Refinement**
  - *hp*— AMR principle and “ingredients”
  - Error evaluation for AMR

► **Application to SFR reactor cores**
  - Benchmarks description and *h*—AMR results
  - Comparison of various *hp*—AMR strategies
  - Energy group dependent *h*—AMR

► **AMR and Perturbation theory**
  - Extension of the standard perturbation formula to spatial discretization effects
  - Possible uses in the AMR framework

► **Summary and perspectives**
Equations system and spatial discretization
- Semi-discrete equations system
- Spatial discretization

hp-Adaptive Mesh Refinement
- hp— AMR principle and “ingredients”
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Summary and perspectives
Semi-discrete equations system

- A spatial domain $\mathcal{D}$ and its boundary $\Gamma$ with $\forall \Omega \in S_2$, $\Gamma = \Gamma_-(\Omega) \cup \Gamma_+(\Omega)$
- Pseudo-stationary neutron transport equation + Multigroup (energy) and $S_N$ (angle) discretizations

Semi-discrete matrix form

$$\begin{align*}
H\phi(\vec{r}) - S\phi(\vec{r}) - \frac{1}{k}F\phi(\vec{r}) &= 0 \quad \text{where} \quad \phi = \left[ \phi_g^G(\vec{r}) = \phi^g(\vec{r}, \Omega_n) \right]_{(g,n)}
\end{align*}$$

with

$$
\begin{align*}
H &= \text{diag} \left[ \vec{\Omega}_n \cdot \vec{\nabla} + \vec{\Sigma}_t^g(\vec{r}) \right]_{(g,n)} \\
S &= \left[ \frac{W_{n'}}{4\pi} \sum_{l=0}^{L} (2l + 1) \sum_{m=-l}^{l} \mathcal{R}_{l,m}(\vec{\Omega}_n) \vec{\Sigma}_{s,l}^{g\rightarrow g'}(\vec{r}) \mathcal{R}_{l,m}(\vec{\Omega}_{n'}) \right]_{(g,n),(g',n')} \\
F &= \left[ \frac{W_{n'}}{4\pi} \chi^g(\vec{r}) \nu \vec{\Sigma}_{f}^g(\vec{r}) \right]_{(g,n),(g',n')}
\end{align*}$$

- a dominant eigenvalue problem with e.g. “void” boundary condition: $\forall g, \forall n, \forall \vec{r} \in \Gamma_-(\Omega_n)$, $\phi^G_n(\vec{r}) = 0$
- Iterative approach to solve the source-flux dependency: classically, three nested iterative loops: e.g. power method (fission) $\supset$ Gauss-Seidel ($g \leftarrow g'$ scattering) $\supset$ Richardson ($g \leftarrow g$ scattering)
Spatial discretization

- Mesh $\mathcal{M}_h = \{\kappa\}$ (structured mesh of lozenges) with cell-wise constant cross-sections
- Discontinuous Galerkin upwind scheme [Reed and Hill, 1973] with non uniform order ($\mathcal{M}_h^p = \{(\kappa, p_\kappa)\}_{\kappa \in \mathcal{M}_h}$)

For any group and direction ($g, n$ indices omitted), the approximation of $\phi$ is considered as

$$\phi_h \in V_h^p = \{ v \in L^2(D) : \forall (\kappa, p_\kappa) \in \mathcal{M}_h^p, v|_\kappa \in Q_{p_\kappa}(\kappa) \}$$

On any $\kappa$, it obeys the following local weak form: $\forall \varphi_h \in V_h^p$,

$$\left< \varphi_h, \left( \Omega \cdot \nabla_h + \Sigma_{t,\kappa} \right) \phi_h \right>_\kappa + \left< \varphi_h^+, \left[ \phi_h \right] \right>_{\partial \kappa}^- = \left< \varphi_h, S \right>_\kappa$$

where
- $\partial \kappa_\pm = \{ \vec{r} \in \partial \kappa : \Omega \cdot n(\vec{r}) \geq 0 \}$
- $\phi_h|_{\partial \kappa}(\vec{r} \in \partial \kappa) = \lim_{s \to 0^\pm} \phi_h(\vec{r} + s\Omega)$
- $\left< f, g \right>_{\partial \kappa}^- = \int_{\partial \kappa^-} d^2r \left| \Omega \cdot n(\vec{r}) \right| f \, g$
- $\left[ \phi_h \right] = (\phi_h^+ - \phi_h^-)$

- special case of stabilization by jump penalization [Brezzi et al., 2004, Brezzi et al., 2006]
- very well suited in this context:
  - direction-dependent cell-by-cell sweeping
  - cell-wise linear system inversion
Spatial discretization

- **Hierarchical polynomial basis** \( \Xi_p(\hat{\kappa}) = \{ f_i \}_{i \in [1, \dim(Q_p(\hat{\kappa}))]} \) s.t. \( \Xi_p(\hat{\kappa}) \subset \Xi_{p+1}(\hat{\kappa}), \forall p \in \mathbb{N} \)

- Adapted from [Shephard et al., 1997], \( \Xi_p(\hat{\kappa}) \) is based on
  - the *topological hierarchy of mesh entities* (vertices, edges, faces and cells) that define the closure of \( \tilde{\kappa} \) of \( \hat{\kappa} \)
  \[
  \tilde{\kappa} = \left\{ M^d_{\hat{\kappa}}, \partial M^d_{\hat{\kappa}} \right\} = \left\{ M^d_{\hat{\kappa}}, M^d_{\hat{\kappa}} \{ M^1_{\hat{\kappa}, j'} \}, \ldots, M^d_{\hat{\kappa}} \{ M^0_{\hat{\kappa}, j'} \} \right\}
  \]
  where \( M^d_{\hat{\kappa}} \{ M^{d'}_{\hat{\kappa}, j'} \} \) is the \( (j')^{th} \) entity of dimension \( d' \) bounding the reference element \( \hat{\kappa} \)
  - with basis functions obtained by *associating shape functions to these entities* s.t. any basis function associated with \( M^{d'}_{\hat{\kappa}, j'} \) vanishes over all “lower order” bounding entities except \( M^{d'}_{\hat{\kappa}, j'} \)
  → of particular interest while *propagating the flux from one element to the other through the boundary trace*

Some functions of \( \Xi_p(\hat{\kappa}) \) over the 2D reference element \([-1, 1] \times [-1, 1]\)

- **node** \( p = 1 \)
- **edge** \( p = 2 \)
- **cell** \( p = 3 \)
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Summary and perspectives
**General objective:** improve in an “optimal” way (vs. number of discretisation dofs, cpu time) the approximate solution accuracy

**Key ingredient:** the (cell-wise) error estimator/indicator
- Error on which quantity of interest?
- How to estimate this error?
- Which criterion to choose between $h-$ or $p-$refinement?

**Additional “choices”** to be made w.r.t.
the *vectorial and iterative nature of the problem*


directed by Raphaèle Herbin from the “Institut de Mathématiques de Marseille”
Error evaluation for AMR: general ideas

▶ Quantity of interest: the \textit{error on the flux} e.g. $\|e\|_{L^2(\kappa)} = \|\phi_h - \phi\|_{L^2(\kappa)}$ (vs. “goal-oriented” method)

▶ Local refinement process: “efficient” evaluation of the \textit{cell-wise error} associated with $\phi_h$
  
  – analysis of an \textit{a posteriori estimator under “minimal” regularity hypotheses} ($\phi \in W^{1,1} \cap L^\infty$)
  
  – “simplifications” in order to get an “efficient” error indicator $E^FV$

▶ “Optimal” choice between $h$-- or $p$--refinement:
  
  – to be naturally related to the “\textit{local}” \textit{regularity} of $\phi$ …
  
  ■ regularity $s$ of $\phi \in H^s$ limited to $\frac{1}{2} - \varepsilon$ or $\frac{3}{2} - \varepsilon$

  ■ loss of regularity in the neighbourhood of characteristic lines directed along $\Omega_n$

  $\rightarrow$ analysis of an \textit{a posteriori} estimator under strong regularity hypotheses ($\phi \in C^\infty$): combined with $E^FV$, $E^R$ as a “\textit{local}” \textit{regularity indicator}

  – …but also to the \textit{convergence regime in $p$} (pre-asymptotic vs. asymptotic) about which \textit{a priori} error estimation (global) results can give useful information

  $\rightarrow$ accordingly, different $hp$-- strategies have been constructed and compared
Global error convergence vs. uniform refinement (order $p$, normalized mesh size $h$)

$$\|e\|_{L^2(D)} = \epsilon_{p,h} = O \left( \frac{h_{\min(p+1,s)}}{p^\alpha} \right)$$

with $s = \frac{1}{2} - \varepsilon$ (MMS0) or $s = \frac{3}{2} - \varepsilon$ (MMS1) (theoretical results e.g. [Houston and Süli, 2001] + numerical tests using manufactured solutions)
**Global error convergence** vs. uniform refinement (order $p$, normalized mesh size $h$)

$$\|e\|_{L^2(D)} = \epsilon_{p,h} = O \left( \frac{h^{\min(p+1,s)}}{p^\alpha} \right) \quad \text{avec } \alpha = \begin{cases} s - \frac{1}{2} & \text{asymptotic} \\ \frac{s}{s} & \text{pre-asymptotic} \end{cases}$$

with $s = \frac{1}{2} - \varepsilon$ (MMS0) or $s = \frac{3}{2} - \varepsilon$ (MMS1)

*(theoretical results)* e.g. [Houston and Süli, 2001] + numerical tests using manufactured solutions

**Refinement “performance”** evaluated through $J_{p \rightarrow p', h \rightarrow h'} = \frac{|\log(\epsilon_{p', h'}) - \log(\epsilon_{p, h})|}{\log(dof_{p', h'}) - \log(dof_{p, h})}$

($dof_{p, h}$, number of degrees of freedom)

**Comparison between $J_{p \rightarrow p+1, h}$ and $J_{p, h \rightarrow h/2}$:**

$p$—refinement is “better” if $\frac{\alpha}{s} \geq \frac{\log \left( \frac{p+2}{p+1} \right)}{\log \left( \frac{p+1}{p} \right)}$

A first important observation to be taken into account in an $hp$—strategy

- in the *pre-asymptotic regime* (resp. asymptotic), $p$—refinement (resp. $h$—refinement)
- numerical tests on MMS0 and MMS1: *pre-asymptotic regime, at least up to $p = 4$*
Two estimators/indicators under very different regularity hypotheses

$E^R$ for $\phi \in C^\infty$ (strong regularity) [Fournier et al., 2011]

Efficiency index for MMS0 in direction $\vec{\Omega} = \frac{1}{\sqrt{2}} (\vec{e}_x + \vec{e}_y)$

Efficiency index for MMS1 in direction $\vec{\Omega} = \frac{1}{\sqrt{2}} (\vec{e}_x + \vec{e}_y)$

Refined mesh obtained in MMS0 case
Error evaluation for AMR: *a posteriori* estimators/indicators

- **Two estimators/indicators** under very *different regularity hypotheses*

\[ E^{FV} \text{ for } \phi \in W^{1,1} \cap L^\infty ("minimal" regularity) [Fournier et al., 2013] \]

- rigorous *a posteriori* estimation based on [Dedner et al., 2007]
- “simplifications” → error indicator: 
  \[ E^{FV} = \sum_{b \in B_h(\kappa)} s_b \int_{b} r^2 |\bar{\Omega} \cdot \bar{n}| \langle \phi_h \rangle \]

→ a “justification” of the heuristic use of such \[ \langle \phi_h \rangle \] -based indicators elsewhere e.g. [Owens et al., 2017]

Refined mesh with \[ E^{FV} - \text{MMS0} \]

Error convergence – refinement driven by \[ E^{FV} \]
or \[ E^R - \text{MMS0} \]
Error evaluation for AMR: *a posteriori* estimators/indicators

- **Two estimators/indicators** under very *different regularity hypotheses*
- **Two “supplementary” error estimators/indicators**
  - $E^R$: a “bad” estimator in our framework but an *indicator of the local regularity*
  - $E^{FV}$: a “good” *error indicator for the cells selection* at each step of the AMR process
Error evaluation for AMR for neutronics equations system

- In all cases, a *same spatial mesh for all directions* $\tilde{\Omega}_n$
  - cell-wise error indicator on the so-called scalar flux (angular flux integrated over $S_2$)
    \[ \tilde{E}^g(\kappa) = \sum_n w_n E_n^g(\kappa) \]
  - the source is “weakly” anisotropic $\rightarrow$ mainly dependent on the scalar flux

- Two different approaches for the *energy groups*:
  - either, a *same spatial mesh for all* energy groups $g$

  \[ \hat{E}(\kappa) = \max_{1 \leq g \leq G} \tilde{E}^g(\kappa) \]
  - refinement criterion:
    \[ \left( \text{cell } \kappa^* \text{ of the unique spatial mesh is refined iff } \hat{E}(\kappa^*) \geq \alpha \max_{\kappa \in \mathcal{M}_h} \hat{E}(\kappa) \right) \quad \text{with } \alpha, \text{ a user parameter} \]

- or *group-wise spatial meshes* associated with “condensed” energy mesh (1 $\leq g \leq G$)

  \[ \hat{E}^g(\kappa) = \max_{g \in C(g)} \tilde{E}^g(\kappa) \]
  - refinement criterion:
    \[ \left( \text{cell } \kappa^* \text{ of the spatial mesh of macro-group } g \text{ is refined iff } \hat{E}^g(\kappa^*) \geq \alpha \max_{\kappa \in \mathcal{M}_h} \hat{E}(\kappa) \right) \quad \text{with } \alpha, \text{ a user parameter} \]
  - $L_2$-projection operators for the construction of the group coupling source terms

- AMR loop “on top” of the power iteration loop for the eigenvalue problem solution
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- Summary and perspectives
Benchmarks description and $h$–AMR results

- **2D and 3D benchmarks** constructed from the ZONA2B core of the CIRANO experimental program (MASURCA, CEA Cadarache)
  - *steel reflector* around the core: important flux variations at the interface with a very strong coupling between the space and energy variations
  - core composed of three homogeneized and “condensed” ($G = 33$ energy groups) materials
    - FUEL (sodium + U/PuO$_2$ fuel)
    - REFLECTOR (3/4 steel + 1/4 sodium)
    - SHIELD (steel)
  - *initial non-conforming mesh coarsening* in order to describe the geometry with a reduced number of cells

![Diagram of mesh coarsening with 529 cells, 34 cells, and 3157 cells]
Benchmarks description and $h$–AMR results

- **2D and 3D benchmarks** constructed from the ZONA2B core of the CIRANO experimental program (MASURCA, CEA Cadarache)
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  - initial non-conforming mesh coarsening in order to describe the geometry with a reduced number of cells

- In the frame of P. Archier’s Ph.D. thesis, $h$–AMR

Adaptive vs. uniform $h$–refinement for $p = 2$ (a same spatial mesh for all groups)

- flux error $\varepsilon_{L^2} = \max_g \sum_n w_n \| \varepsilon_n^g \|_{L^2(D)}$
  - w.r.t to a reference solution with a very fine mesh and high order $p$

- in 2D: flux error of $10^{-3}$
  - computational time $\times 1/4$

- in 3D: computational time of 5000s
  - flux error $\times 1/4$
Comparison of various $hp$—AMR strategies

- Comparison of *different $hp$—strategies* for choosing between $h$— and $p$—refinement (2D case)

  - AMR cells selection criterion $E^{FV}(\kappa) > \alpha$, a same spatial mesh for all groups, with

<table>
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<th>notation</th>
<th>strategy</th>
<th>$h$—refinement criterion</th>
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<tr>
<td>$hp^B$</td>
<td><em>a priori</em> observation</td>
<td>$p_\kappa &gt; 4$</td>
</tr>
<tr>
<td>$hp^{2E}$</td>
<td>two estimators</td>
<td>$E^R(\kappa) \leq \alpha$</td>
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<tr>
<td>$hp_{mod}^{2E}$</td>
<td>two estimators + <em>a priori</em> observation</td>
<td>$E^R(\kappa) \leq \alpha$ and $p_\kappa &gt; 4$</td>
</tr>
<tr>
<td>$hp^{TP}$</td>
<td>“type-parameter” [Gui and Babuška, 1986]</td>
<td>$E^F_p(\kappa)/E^F_{p-1}(\kappa) \leq \alpha$</td>
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*Main criterion for $h$— or $p$—refinement: pre-asymptotic/asymptotic $p$ convergence regime*
Energy group dependent $h$–AMR

- $h$–AMR on the two extreme cases: one single mesh or $G = 33$ different meshes
- $E^{FV}$ and $E^R$ compared for “driving” the adaptation process
- both $\varepsilon_{L^2}$ and $|k - k_h|$ (eigenvalue error) monitored

$h$-refinement for $p = 1$ driven by either $E^R$ or $E^{FV}$

- w.r.t. $\varepsilon_{L^2}$: $E^R$ and $E^{FV}$ gives similar results for 1 mesh; clear difference with $G$ meshes: $E^R$ insufficient; additional dof reduction by a factor of 2 to 3 with $E^{FV}$

- w.r.t. $|k - k_h|$: “counter-performance” with $G$ meshes, consistent with results reported in [Goffin et al., 2013] where it is attributed to the interpolation error associated with the fission source → what about a $k$-oriented AMR?
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- Summary and perspectives
Perturbation theory

A "classical" tool in reactor physics to *decompose the variations of a quantity of interest* (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the *local variations of the parameters* associated with the core configuration (nuclear data, material spatial distribution).
Perturbation theory

A “classical” tool in reactor physics to decompose the variations of a quantity of interest (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the local variations of the parameters associated with the core configuration (nuclear data, material spatial distribution).

▶ “Standard” perturbation formula: $\rho$ variation from configuration $i$ to configuration $p$

In semi-discrete form

$$\Delta \rho \Sigma = - \frac{\left\langle \phi^*_i, \left( \Delta H - \Delta S - \frac{1}{k_p} \Delta F \right) \phi_p \right\rangle_{DG \times N}}{\left\langle F^* \phi^*_i, \phi_p \right\rangle_{DG \times N}}$$

$$\left\langle \phi, \psi \right\rangle_{DG \times N} = \sum_{g=1}^{G} \sum_{n=1}^{N} W_n \left\langle \phi^G_n, \psi^G_n \right\rangle_D$$

- $\phi^*_i$ is the adjoint flux in configuration $i$ and, in particular $\Delta H = \text{diag} \left[ \Delta \bar{\Sigma}_t^g (R) \right]_{(g,n)}$

→ valid only if same discretization for both $\phi^*_i$ and $\phi_p$ i.e. upwind DG: $\phi_i, \phi_p \in (V_h^p)^{G \times N}$
A “classical” tool in reactor physics to decompose the variations of a quantity of interest (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the local variations of the parameters associated with the core configuration (nuclear data, material spatial distribution).

- “Standard” perturbation formula: $\rho$ variation from configuration $i$ to configuration $p$
- In the DG framework, this formula extended as $\Delta \rho_{hh'} = \Delta \rho_{\Sigma}^{hh'} + \Delta \rho_{MM}^{hh'}$ [Le Tellier et al., 2011]
- an extra term $\Delta \rho_{MM}^{hh'}$ when $\phi_i \in (V_h^p)^{G \times N}$ and $\phi_p \in (V_{h'}^p)^{G \times N}$
- in particular, if $V_h^p$ is a refinement of $V_{h'}^p$, it can be written as:

$$\Delta \rho_{hh'} = - \frac{\left\langle \pi_{h'} \phi_{i,h'}^*, \phi_{p,h'} \right\rangle_{B_{h'}^{G \times N}} - \left\langle \phi_{i,h}^*, \pi_h \phi_{p,h'} \right\rangle_{B_{h}^{G \times N}}}{\left\langle F_{i,h}^*, \phi_{i,h'}^*, \phi_{p,h'} \right\rangle_{D^{G \times N}}}$$

- $\pi_h$ (resp. $\pi_{h'}$), the $L^2$-projector on $V_h^p$ (resp. $V_{h'}^p$),
- $\vec{\nabla}_{hh'}$, the broken gradient operator on the union mesh between $\mathcal{M}_h$ and $\mathcal{M}_{h'}$,
- $\left\langle \varphi_{h}, \phi_{h} \right\rangle_{B_{h}^{G \times N}} = \sum_{g=1}^{G} \sum_{n=1}^{N} w_n \sum_{\kappa \in \mathcal{M}_h} \left\langle \varphi_{h,n}, \left[ \phi_{h,n}^g \right]_{\partial \kappa} \right\rangle_{\partial \kappa}$. 

Romain Le Tellier
Perturbation theory

a “classical” tool in reactor physics to decompose the variations of a quantity of interest (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the local variations of the parameters associated with the core configuration (nuclear data, material spatial distribution)

- “Standard” perturbation formula: $\rho$ variation from configuration $i$ to configuration $p$
- In the DG framework, this formula extended as $\Delta \rho_{hh'} = \Delta \rho^{hh'}_{\Sigma} + \Delta \rho^{hh'}_{\mathcal{M}}$ [Le Tellier et al., 2011]
- a priori, this extended formula can be practically useful in two different ways...
Evaluating the reactivity effect associated with a modification of the “physical” parameters:

*Improve the AMR performance when evaluating a “physical” reactivity change*

Geometry and core materials

- **Group 1 (fast)**
- **Group 2 (thermal)**

Scalar fluxes – configuration “rod in”

*For AMR to be fully interesting when evaluating a reactivity worth, $\Delta \rho_{\Sigma}^{hh'}$ to be used instead of $\Delta \rho_{\Sigma}^{hh}$*

Error on the rod reactivity “worth” – $h$-refinement

- “direct-driven” (DD): AMR for direct problem and meshes imposed for adjoint problem
- “adjoint-driven” (AD): AMR for adjoint problem and meshes imposed for adjoint problem
- “configuration-dependent” (CD): independent AMR for direct and adjoint problems
Evaluating the change in reactivity due to a mesh refinement:

Towards a “\( \rho \)-oriented” mesh adaptation?

- “two-mesh” error estimators: \( E^{2M} \) (flux difference) and \( E^{2M}_\rho (\Delta \rho^{hh'}) \) decomposition over the mesh
- \( h \)-refinement on the ZONA2B benchmark (2D case) with different estimators

No significant gain on \( \rho \) convergence when using a \( \rho \) error estimator vs. a flux error estimator

Consistent with the results reported in [Owens et al., 2017] where the more standard “Dual Weighted Residual” approach (firstly used in neutronics in [Lathouwers, 2011]) is used
Summary

- “Sodium Fast Reactors” core physics, ASTRID industrial project
- Investigation of $hp$—AMR for reducing the burden of solving the neutron transport (Multigroup, $S_N$, DG upwing scheme)
- Two different error estimators/indicators under very different regularity hypotheses
- Importance of the pre-asymptotic/asymptotic p—convergence regime on dofs reduction
- Connecting the classical “perturbation theory” in reactor core physics with AMR
Summary

▶ “Sodium Fast Reactors” core physics, ASTRID industrial project
▶ Investigation of $hp$—AMR for reducing the burden of solving the neutron transport ($\text{Multigroup, } S_N$, $\text{DG upwing scheme}$)
▶ Two different error estimators/indicators under very different regularity hypotheses
▶ Importance of the pre-asymptotic/asymptotic $p$—convergence regime on dofs reduction
▶ Connecting the classical “perturbation theory” in reactor core physics with AMR

And beyond

▶ AMR with DG upwind on polygonal meshes
  (using “non polynomial” element bases $\rightarrow$ D. Labeurthre ongoing Ph.D. thesis with A. Calloo)
▶ Performance optimization:
  – Acceleration of these nested iterative loops by a simplified operator (e.g. diffusion) based
  preconditioning (so-called “Diffusion Synthetic acceleration”)
  – Nested iterative loops for solving the source-flux dependency $\rightarrow$ where to put the AMR loop?
▶ Energy-space coupling:
  – “Adaptive” condensed energy mesh for group-wise spatial mesh adaptation
  – Spatial domain decomposition with subdomain-dependent energy meshes
Thank you for your attention

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Stabilization mechanisms in discontinuous Galerkin finite element methods.

Discontinuous Galerkin methods for first-order hyperbolic problems.

Publisher: Society for Industrial and Applied Mathematics.

Discontinuous Galerkin discretization and hp-refinement for the resolution of the neutron transport equation.

Analysis of an a posteriori error estimator for the transport equation with $S_n$ and discontinuous Galerkin discretizations.

Minimising the error in eigenvalue calculations involving the boltzmann transport equation using goal-based adaptivity on unstructured meshes.

The h, p and h-p versions of the finite element method in 1 dimension. part III. the adaptive h-p version.
References

hp — adaptive discontinuous Galerkin finite element methods for first-order hyperbolic problems. 

Goal-oriented spatial adaptivity for the sn equations on unstructured triangular meshes.

Reactivity perturbation formulation for a discontinuous Galerkin based transport solver and its use with adaptive mesh refinement.

Energy dependent mesh adaptivity of discontinuous isogeometric discrete ordinate methods with dual weighted residual error estimators.
Journal of Computational Physics, 335:352 – 386.

A two-mesh adaptive mesh refinement technique for Sn neutral-particle transport using a higher-order DGFEM.

Triangular mesh methods for neutron transport equation.

A straightforward structure to construct shape functions for variable p — order meshes.