hp—AMR applied to the neutron transport equation discretized by a Discontinuous Galerkin upwind scheme

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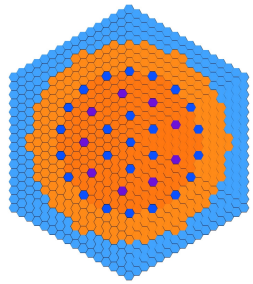
Context

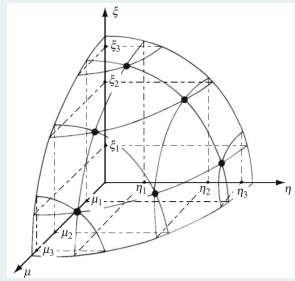


- *Nuclear reactor core physics* (see F. Madiot's presentation)
- For "*Sodium Fast Reactors*" (SFR), the diffusion (or SP_N) approximation is deemed insufficient for *core design calculations* (homogenized assemblies)
- In the early stage (2009-2012) of the *ASTRID prototype* (cancelled in 2019) design studies, development of a *3D neutron transport solver* to supplement the existing ERANOS code

SNATCH solver

Along the following lines:

- angular discretization: so-called S_N *method* (quadrature-based): $\vec{\Omega} \in S_2 \rightarrow \left(w_n, \vec{\Omega}_n\right)_{1 \leq n \leq N}$
- high-order spatial scheme $\rightarrow$ *Discontinuous Galerkin (DG) upwind scheme*
- both direct and adjoint solutions $\rightarrow$ *perturbation theory toolbox*



With the additional idea of having a modular environment to test numerical methods

- In this framework, *AMR has been investigated*

This presentation

the salient features of this (old) work + "links" to more recent works by other researchers on this subject

Contents

- **Equations system and spatial discretization**
  - Semi-discrete equations system
  - Spatial discretization
- ***hp*-Adaptive Mesh Refinement**
  - *hp*— AMR principle and “ingredients”
  - Error evaluation for AMR
- **Application to SFR reactor cores**
  - Benchmarks description and *h*—AMR results
  - Comparison of various *hp*—AMR strategies
  - Energy group dependent *h*—AMR
- **AMR and Perturbation theory**
  - Extension of the standard perturbation formula to spatial discretization effects
  - Possible uses in the AMR framework
- **Summary and perspectives**

Contents

Semi-discrete equations system

- A spatial domain $\mathcal{D}$ and its boundary Γ with $\forall \vec{\Omega} \in S_2$, $\Gamma = \Gamma_-(\vec{\Omega}) \dot{\cup} \Gamma_+(\vec{\Omega})$
- *Pseudo-stationary* neutron transport equation + *Multigroup* (energy) and S_N (angle) discretizations

Semi-discrete matrix form

$$\underline{\underline{H}}\,\underline{\phi}(\vec{r}) - \underbrace{\underline{\underline{S}}\,\underline{\phi}(\vec{r}) - \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}(\vec{r})}_{\underline{S}(\vec{r})} = 0 \quad \text{where} \quad \underline{\phi} = \left[\phi_n^g(\vec{r}) = \phi^g(\vec{r}, \vec{\Omega}_n)\right]_{(g,n)}$$

with

$$\begin{aligned}
\underline{\underline{H}} &= \operatorname{diag}\left[\vec{\Omega}_n \cdot \vec{\nabla} + \bar{\Sigma}_t^g(\vec{r})\right]_{(g,n)} \\
\underline{\underline{S}} &= \left[\frac{w_{n'}}{4\pi} \sum_{l=0}^{L} (2l+1) \sum_{m=-l}^{l} \mathcal{R}_{l,m}(\vec{\Omega}_n) \bar{\Sigma}_{s,l}^{g \leftarrow g'}(\vec{r}) \mathcal{R}_{l,m}(\vec{\Omega}_{n'})\right]_{(g,n),(g',n')} \\
\underline{\underline{F}} &= \left[\frac{w_{n'}}{4\pi} \chi^g(\vec{r}) \nu \Sigma_f^{g'}(\vec{r})\right]_{(g,n),(g',n')}
\end{aligned}$$

- a *dominant eigenvalue problem* with e.g. "void" boundary condition: $\forall g, \forall n, \forall \vec{r} \in \Gamma_-(\vec{\Omega}_n)$, $\phi_n^g(\vec{r}) = 0$
- *Iterative approach* to solve the *source-flux dependency*: classically, three nested iterative loops: e.g. power method (fission) $\supset$ Gauss-Seidel ($g \leftarrow g'$ scattering) $\supset$ Richardson ($g \leftarrow g$ scattering)

Spatial discretization

- Mesh $\mathcal{M}_h = \{\kappa\}$ (*structured mesh of lozenges*) with cell-wise constant cross-sections
- *Discontinuous Galerkin upwind scheme* [Reed and Hill, 1973] with non uniform order ($\mathcal{M}_h^{\mathbf{p}} = \{(\kappa, p_\kappa)\}_{\kappa \in \mathcal{M}_h}$)

For any group and direction (g, n indices omitted), the approximation of ϕ is considered as

$$\phi_h \in V_h^{\mathbf{p}} = \left\{ v \in L^2(\mathcal{D}) : \forall (\kappa, p_\kappa) \in \mathcal{M}_h^{\mathbf{p}}, v|_\kappa \in Q_{p_\kappa}(\kappa) \right\}$$

On any κ, it obeys the following *local weak form*: $\forall \varphi_h \in V_h^{\mathbf{p}}$,

$$\left\langle \varphi_h, \left(\vec{\Omega} \cdot \vec{\nabla}_h + \Sigma_{t,\kappa} \right) \phi_h \right\rangle_\kappa + \left\langle \varphi_h^+, [\![\phi_h]\!] \right\rangle_{\partial\kappa_-} = \left\langle \varphi_h, S \right\rangle_\kappa$$

where

- $\partial\kappa_\pm = \left\{ \vec{r} \in \partial\kappa : \vec{\Omega} \cdot \vec{n}(\vec{r}) \gtrless 0 \right\}$
- $\phi_h|^\pm_{\partial\kappa}(\vec{r} \in \partial\kappa) = \lim_{s \to 0^\pm} \phi_h \left(\vec{r} + s\vec{\Omega} \right)$
- $\left\langle f, g \right\rangle_{\partial\kappa_-} = \int_{\partial\kappa_-} d^2 r \left| \vec{\Omega} \cdot \vec{n}(\vec{r}) \right| f\, g$
- $[\![\phi_h]\!] = \left(\phi_h^+ - \phi_h^- \right)$

- special case of stabilization by *jump penalization* [Brezzi et al., 2004, Brezzi et al., 2006]
- very well suited in this context:
  - *direction-dependent cell-by-cell sweeping*
  - *cell-wise linear system inversion*

Spatial discretization

- *Hierarchical polynomial basis* $\Xi_p(\hat{\kappa}) = \{f_i\}_{i\in[1,\dim(Q_p(\hat{\kappa}))]}$ s.t. $\Xi_p(\hat{\kappa}) \subset \Xi_{p+1}(\hat{\kappa})$, $\forall p \in \mathbb{N}$
- Adapted from [Shephard et al., 1997], $\Xi_p(\hat{\kappa})$ is based on
  - the *topological hierarchy of mesh entities* (vertices, edges, faces and cells) that define the closure of $\bar{\hat{\kappa}}$ of $\hat{\kappa}$

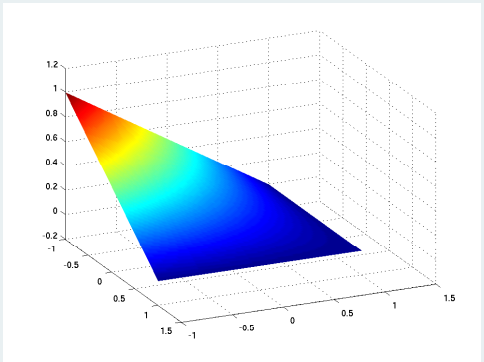
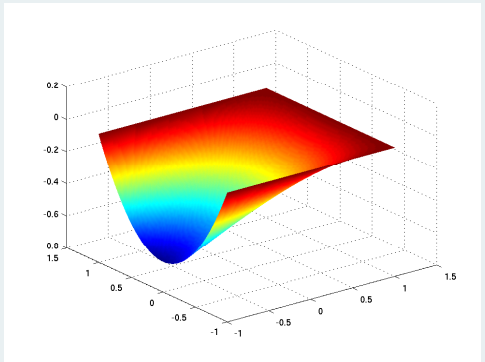
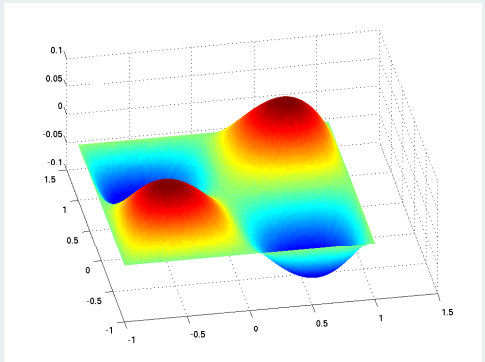
$$\bar{\hat{\kappa}} = \left\{M^d_{\hat{\kappa}}, \partial M^d_{\hat{\kappa}}\right\} = \left\{M^d_{\hat{\kappa}}, M^d_{\hat{\kappa}}\{M^1_{\hat{\kappa},j'}\}, \ldots, M^d_{\hat{\kappa}}\{M^0_{\hat{\kappa},j'}\}\right\}$$

    where $M^d_{\hat{\kappa}}\{M^{d'}_{\hat{\kappa},j'}\}$ is the $(j')^{\text{th}}$ entity of dimension d' bounding the reference element $\hat{\kappa}$
  - with basis functions obtained by *associating shape functions to these entities* s.t. any basis function associated with $M^{d'}_{\hat{\kappa},j'}$ vanishes over all "lower order" bounding entities except $M^{d'}_{\hat{\kappa},j'}$

  $\rightarrow$ of particular interest while *propagating the flux from one element to the other through the boundary trace*

Some functions of $\Xi_p(\hat{\kappa})$ over the 2D reference element $[-1,1]\times[-1,1]$

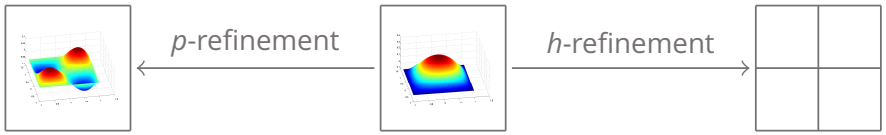
node $p = 1$ | edge $p = 2$ | cell $p = 3$

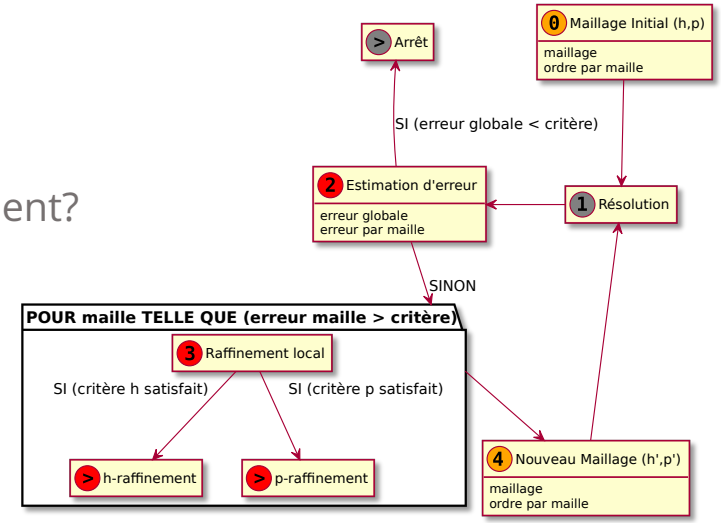
Contents

hp— AMR principle and "ingredients"

- *General objective*: improve in an "optimal" way (vs. number of discretisation dofs, cpu time) the approximate solution accurary



- *Key ingredient*: the (cell-wise) *error estimator/indicator*
  - Error on which quantity of interest?
  - How to estimate this error?
  - Which criterion to choose between *h*— or *p*—refinement?
- Additional "choices" to be made w.r.t. the *vectorial and iterative nature of the problem*



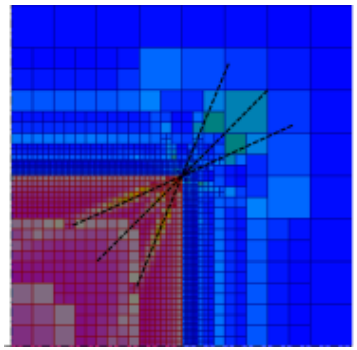
- Back in 2008, little interest in neutronics : S_n – DG – 2D triangular meshes [Ragusa and Wang, 2010, Wang and Ragusa, 2011]

Damien Fournier's Ph.D. thesis (2008-2011)

directed by Raphaèle Herbin from the "Institut de Mathématiques de Marseille"

Error evaluation for AMR: general ideas

- Quantity of interest: the *error on the flux* e.g. $\|e\|_{L^2(\kappa)} = \|\phi_h - \phi\|_{L^2(\kappa)}$ (vs. "goal-oriented" method)
- Local refinement process: "efficient" evaluation of the *cell-wise error* associated with ϕ_h
  - analysis of an *a posteriori estimator under "minimal" regularity hypotheses* ($\phi \in W^{1,1} \cap L^\infty$)
  - "simplifications" in order to get an "efficient" error indicator E^{FV}
- "Optimal" choice between *h— or p—refinement*:
  - to be naturally related to the *"local" regularity* of ϕ ...
    - regularity s of ϕ ($\in H^s$) limited to $\frac{1}{2} - \varepsilon$ or $\frac{3}{2} - \varepsilon$
    - loss of regularity in the neighbourhood of characteristic lines directed along $\vec{\Omega}_n$



$\rightarrow$ analysis of an *a posteriori* estimator under strong regularity hypotheses ($\phi \in C^\infty$): combined with E^{FV}, E^R *as a "local" regularity indicator*

  - ...but also to the *convergence regime in p* (pre-asymptotic vs. asymptotic) about which *a priori* error estimation (global) results can give useful information

$\rightarrow$ *accordingly, different hp— strategies have been constructed and compared*

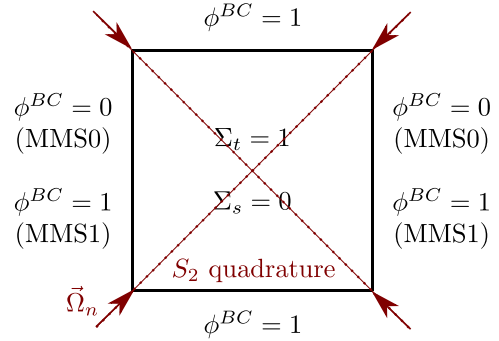
Error evaluation for AMR: *a priori* error estimation

- *Global error convergence* vs. uniform refinement (order p, normalized mesh size h)

$$\|e\|_{L^2(\mathcal{D})} = \epsilon_{p,h} = \mathcal{O}\left(\frac{h^{\min(p+1,s)}}{p^{\alpha}}\right)$$

with $s = \frac{1}{2} - \varepsilon$ (MMS0) or $s = \frac{3}{2} - \varepsilon$ (MMS1)

(*theoretical results* e.g. [Houston and Süli, 2001] + numerical tests using *manufactured solutions*)



Error evaluation for AMR: *a priori* error estimation

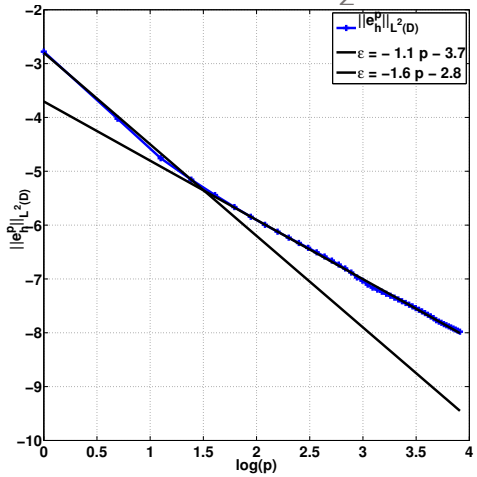
- *Global error convergence* vs. uniform refinement (order p, normalized mesh size h)

$$\|e\|_{L^2(\mathcal{D})} = \epsilon_{p,h} = \mathcal{O}\left(\frac{h^{\min(p+1,s)}}{p^{\alpha}}\right) \text{ avec } \alpha = \begin{cases} s - \frac{1}{2} & \text{asymptotic} \\ s & \text{pre-asymptotic} \end{cases}$$

with $s = \frac{1}{2} - \varepsilon$ (MMS0) or $s = \frac{3}{2} - \varepsilon$ (MMS1)

(*theoretical results* e.g. [Houston and Süli, 2001] + numerical tests using *manufactured solutions*)

MMS1 ($s = \frac{3}{2}$)



- *Refinement "performance"* evaluated through $J_{p\to p',h\to h'} = \dfrac{\left|\log(\epsilon_{p',h'}) - \log(\epsilon_{p,h})\right|}{\log(dof_{p',h'}) - \log(dof_{p,h})}$

  ($dof_{p,h}$, number of degrees of freedom)

- Comparison between $J_{p\to p+1,h}$ and $J_{p,h\to h/2}$: $\left(p-\text{refinement is "better" if } \frac{\alpha}{s} \geq \dfrac{\log\left(\frac{p+2}{p+1}\right)}{\log\left(\frac{p+1}{p}\right)}\right)$

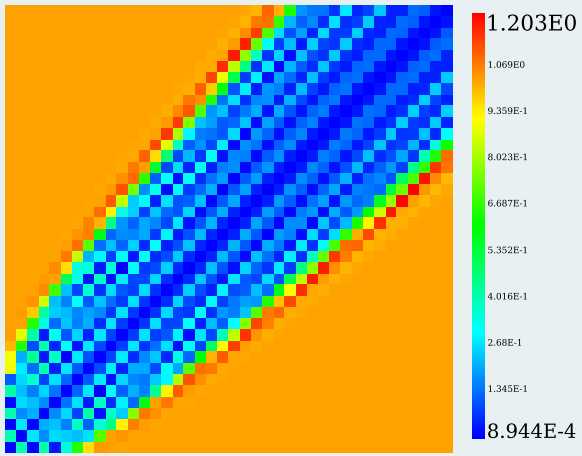
A first important observation to be taken into account in an *hp*−strategy

- in the *pre-asymptotic regime* (resp. asymptotic), *p−refinement* (resp. *h*−refinement)
- numerical tests on MMS0 and MMS1: *pre-asymptotic regime, at least up to* $p = 4$

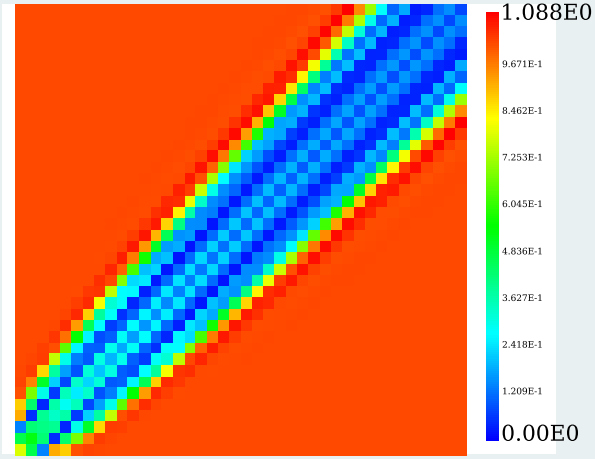
Error evaluation for AMR: *a posteriori* estimators/indicators

- *Two estimators/indicators* under very *different regularity hypotheses*

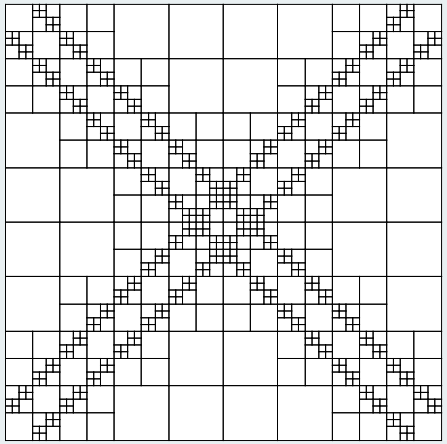
E^R for $\phi \in C^\infty$ (*strong regularity*) [Fournier et al., 2011]



Efficiency index for MMS0 in direction $\vec{\Omega} = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_y)$



Efficiency index for MMS1 in direction $\vec{\Omega} = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_y)$



Refined mesh obtained in MMS0 case

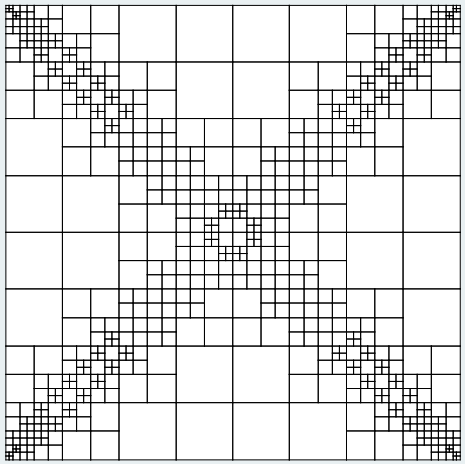
Error evaluation for AMR: *a posteriori* estimators/indicators

- *Two estimators/indicators* under very *different regularity hypotheses*

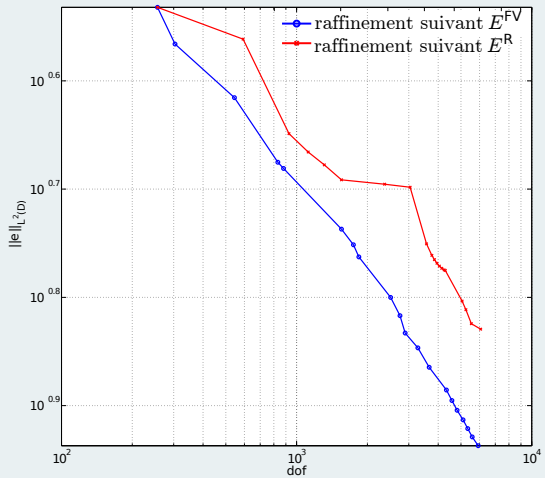
E^{FV} for $\phi \in W^{1,1} \cap L^\infty$ (*"minimal" regularity*) [Fournier et al., 2013]

- rigorous *a posteriori* estimation based on [Dedner et al., 2007]
- "simplifications" $\rightarrow$ error indicator: $E^{FV} = \sum_{b \in \mathcal{B}_h(\kappa)} S_b \int_b d^2 r \left| \vec{\Omega} \cdot \vec{n} \right| [\![\phi_h]\!]$

$\rightarrow$ a "justification" of the heuristic use of such $[\![\phi_h]\!]$-based indicators elsewhere e.g. [Owens et al., 2017]



Refined mesh with E^{FV} - MMS0



Error convergence – refinement driven by E^{FV} or E^{R} – MMS0

Error evaluation for AMR: *a posteriori* estimators/indicators

- *Two estimators/indicators* under very *different regularity hypotheses*
- *Two "supplementary" error estimators/indicators*
  - E^R : a "bad" estimator in our framework but an *indicator of the local regularity*
  - E^{FV} : a "good" *error indicator for the cells selection* at each step of the AMR process

Error evaluation for AMR for neutronics equations system

- In all cases, a *same spatial mesh for all directions* $\vec{\Omega}_n$
  - cell-wise error indicator on the so-called scalar flux (angular flux integrated over S_2)
    $\bar{E}^g(\kappa) = \sum_n w_n E_n^g(\kappa)$
  - the source is "weakly" anisotropic $\rightarrow$ mainly dependent on the scalar flux
- Two different approaches for the *energy groups*:
  - either, a *same spatial mesh for all* energy groups g

- $\hat{E}(\kappa) = \max_{1 \leq g \leq G} \bar{E}^g(\kappa)$
- refinement criterion $\left(\begin{array}{c} \text{cell } \kappa^\star \text{ of the unique spatial mesh is refined iff} \\ \hat{E}(\kappa^\star) \geq \alpha \max_{\kappa \in \mathcal{M}_h} \hat{E}(\kappa) \end{array} \right)$ with α, a user parameter

  - or *group-wise spatial meshes* associated with "condensed" energy mesh ($1 \leq \mathbb{g} \leq \mathbb{G}$)

- $\hat{E}^{\mathbb{g}}(\kappa) = \max_{g \in \mathcal{C}(\mathbb{g})} \bar{E}^g(\kappa)$
- refinement criterion: $\left(\begin{array}{c} \text{cell } \kappa^\star \text{ of the spatial mesh of macro-group } \mathbb{g} \text{ is refined iff} \\ \hat{E}^{\mathbb{g}}(\kappa^\star) \geq \alpha \max_{\kappa \in \mathcal{M}_h} \hat{E}(\kappa) \end{array} \right)$ with α, a user parameter
- L_2-projection operators for the construction of the group coupling source terms

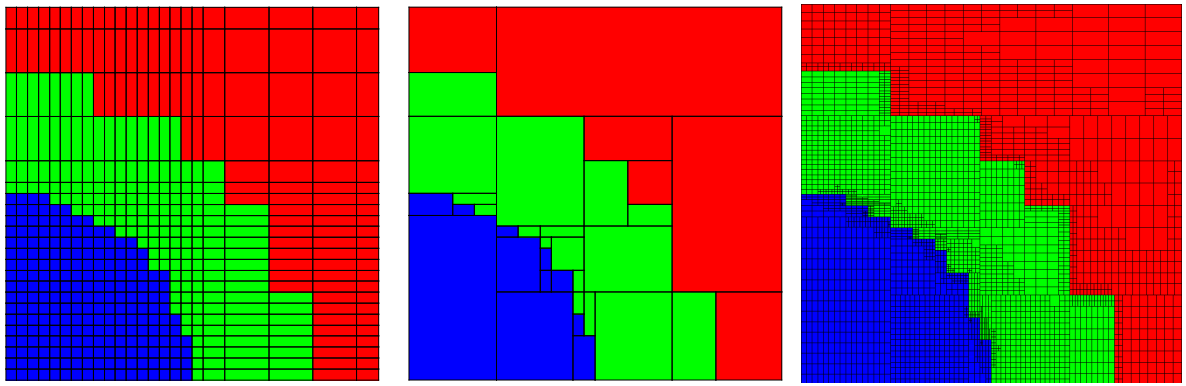
- AMR loop "on top" of the power iteration loop for the eigenvalue problem solution

Contents

Benchmarks description and $h-$AMR results

- *2D and 3D benchmarks* constructed from the *ZONA2B core* of the CIRANO experimental program (MASURCA, CEA Cadarache)

  - *steel reflector* around the core: important flux variations at the interface with a very strong coupling between the space and energy variations
  - core composed of three homogeneized and "condensed" (*$G = 33$ energy groups*) materials
    - FUEL (sodium + U/PuO_2 fuel)
    - REFLECTOR (3/4 steel + 1/4 sodium)
    - SHIELD (steel)
  - *initial non-conforming mesh coarsening* in order to describe the geometry with a reduced number of cells



529 cells | 34 cells | 3157 cells

Benchmarks description and $h-$AMR results

- *2D and 3D benchmarks* constructed from the *ZONA2B core* of the CIRANO experimental program (MASURCA, CEA Cadarache)

  - *steel reflector* around the core: important flux variations at the interface with a very strong coupling between the space and energy variations
  - core composed of three homogeneized and "condensed" (*G =33 energy groups*) materials
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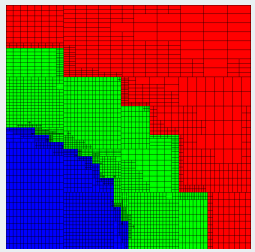
  - *initial non-conforming mesh coarsening* in order to describe the geometry with a reduced number of cells

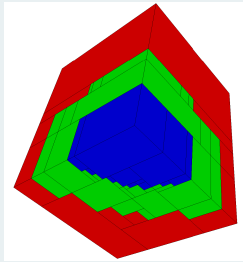
- In the frame of P. Archier's Ph.D. thesis, *$h-$ AMR*

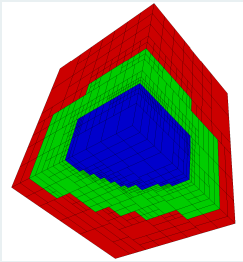
Adaptive vs. uniform *$h-$refinement for $p = 2$* (a same spatial mesh for all groups)

- flux error $\varepsilon_{L^2} = \max_g \sum_n w_n \left\| e_n^g \right\|_{L^2(\mathcal{D})}$

  w.r.t to a reference solution with a very fine mesh and high order p





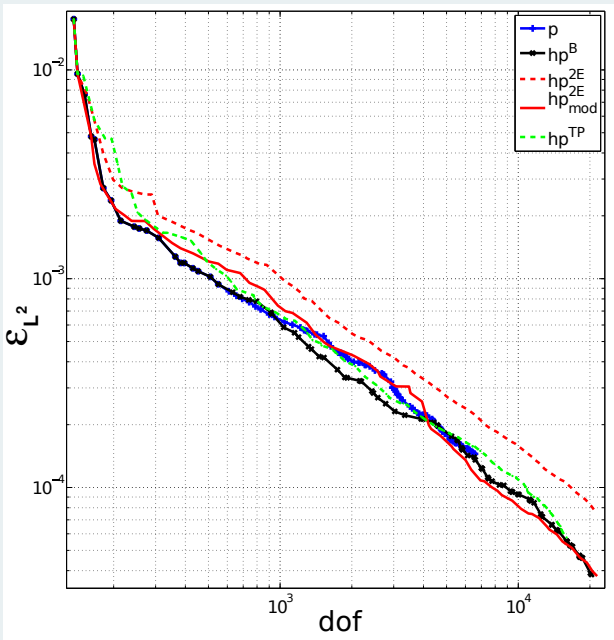


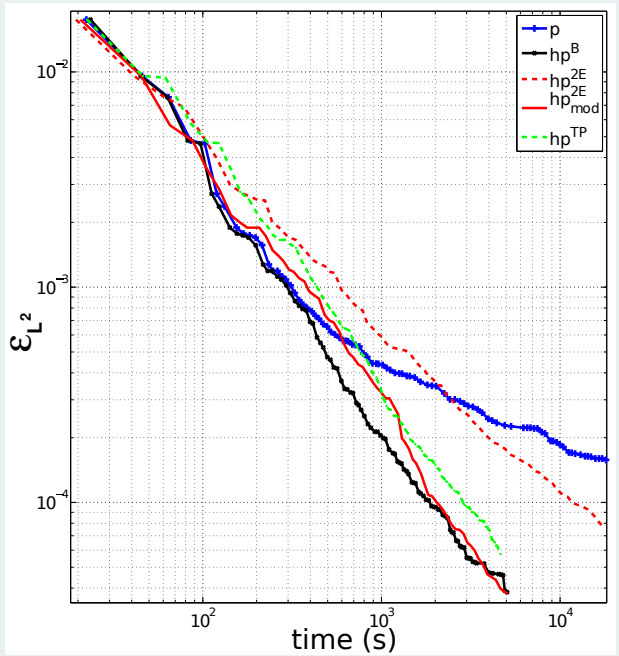
- in 2D: flux error of 10^{-3}
  $\rightarrow$ *computational time* $\times 1/4$
- in 3D: computational time of 5000s
  $\rightarrow$ *flux error* $\times 1/4$

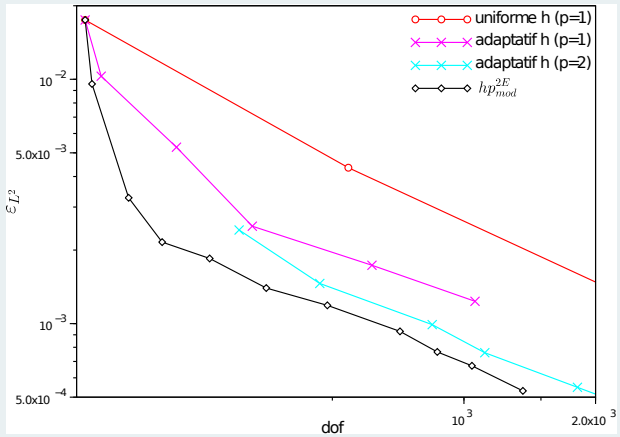
Comparison of various *hp*—AMR strategies

- Comparison of *different hp— strategies* for choosing between *h*— and *p*—refinement (2D case)
  - AMR cells selection criterion $E^{FV}(\kappa) > \alpha$, a same spatial mesh for all groups, with

notation	strategy	*h*—refinement criterion
hp^{B}	*a priori* observation	$p_\kappa > 4$
hp^{2E}	two estimators	$E^{R}(\kappa) \leq \alpha$
hp^{2E}_{mod}	two estimators + *a priori* observation	$E^{R}(\kappa) \leq \alpha$ and $p_\kappa > 4$
hp^{TP}	"type-parameter" [Gui and Babuška, 1986]	$E^{FV}_{p}(\kappa)/E^{FV}_{p-1}(\kappa) \leq \alpha$





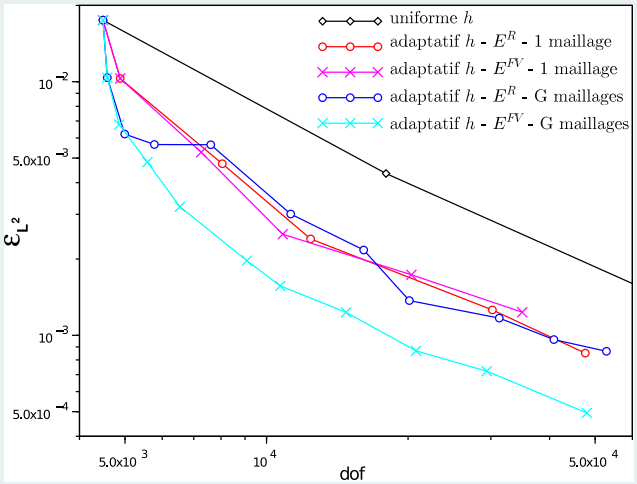


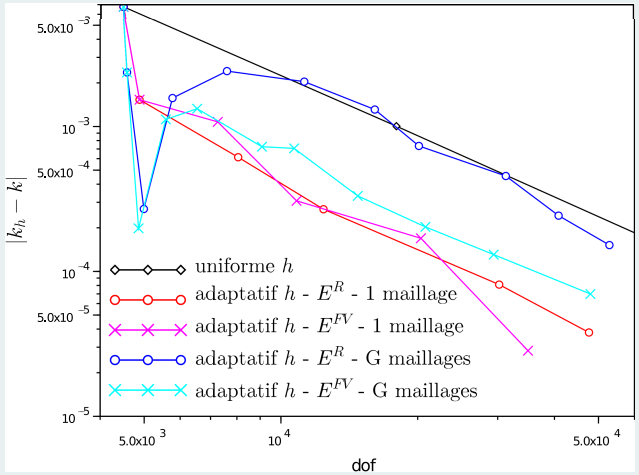
a reduction factor of at least 2 between h—AMR p = 2 and hp—AMR on dofs

- *Main criterion for h— or p—refinement: pre-asymptotic/asymptotic p convergence regime*

Energy group dependent $h-$AMR

- $h-AMR$ on the two extreme cases: *one single mesh* or $G = 33$ *different meshes*
- E^{FV} and E^{R} compared for "driving" the adaptation process
- both ε_{L^2} and $|k - k_h|$ (eigenvalue error) monitored





h-refinement for $p = 1$ driven by either E^R or E^{FV}

- w.r.t. ε_{L^2} : E^R and E^{FV} gives similar results for 1 mesh ; clear difference with G meshes: E^R insufficient; *additional dof reduction by a factor of 2 to 3 with* E^{FV}
- w.r.t. $|k - k_h|$: "counter-performance" with G meshes, consistent with results reported in [Goffin et al., 2013] where it is attributed to the interpolation error associated with the fission source $\rightarrow$ *what about a k-oriented AMR?*

Contents

- **Equations system and spatial discretization**
  - Semi-discrete equations system
  - Spatial discretization
- ***hp*-Adaptive Mesh Refinement**
  - *hp*— AMR principle and “ingredients”
  - Error evaluation for AMR
- **Application to SFR reactor cores**
  - Benchmarks description and *h*—AMR results
  - Comparison of various *hp*—AMR strategies
  - Energy group dependent *h*—AMR
- **AMR and Perturbation theory**
  - Extension of the standard perturbation formula to spatial discretization effects
  - Possible uses in the AMR framework
- **Summary and perspectives**

AMR and Perturbation theory

Perturbation theory

a “classical” tool in reactor physics to *decompose the variations of a quantity of interest* (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the *local variations of the parameters* associated with the core configuration (nuclear data, material spatial distribution)

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- *"Standard" perturbation formula*: *ρ variation* from configuration *i* to configuration *p*

In semi-discrete form

$$\Delta\rho_{\Sigma} = -\frac{\left\langle \underline{\phi}_i^{\star}, \left(\Delta\underline{\underline{H}} - \Delta\underline{\underline{S}} - \frac{1}{k_p}\Delta\underline{\underline{F}}\right)\underline{\phi}_p \right\rangle_{\mathcal{D}^{G\times N}}}{\left\langle \underline{\underline{F}}_i^{\star}\underline{\phi}_i^{\star}, \underline{\phi}_p \right\rangle_{\mathcal{D}^{G\times N}}}$$

- $\left\langle \underline{\phi}, \underline{\psi} \right\rangle_{\mathcal{D}^{G\times N}} = \sum_{g=1}^{G}\sum_{n=1}^{N} w_n \left\langle \phi_n^g, \psi_n^g \right\rangle_{\mathcal{D}}$
- $\underline{\phi}_i^{\star}$ is the adjoint flux in configuration *i* and, in particular $\Delta\underline{\underline{H}} = \mathrm{diag}\left[\Delta\bar{\Sigma}_t^g(\vec{r})\right]_{(g,n)}$

$\rightarrow$ valid only if same discretization for both $\underline{\phi}_i^{\star}$ and $\underline{\phi}_p$ *i.e.* upwind DG: $\underline{\phi}_i, \underline{\phi}_p \in (V_h^{\mathbf{p}})^{G\times N}$

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- *"Standard" perturbation formula*: ρ *variation* from configuration i to configuration p
- In the DG framework, this *formula extended* as $\Delta\rho^{hh'} = \Delta\rho^{hh'}_{\Sigma} + \Delta\rho^{hh'}_{\mathcal{M}}$ [Le Tellier et al., 2011]

- an extra term $\Delta\rho^{hh'}_{\mathcal{M}}$ when $\underline{\phi}_i \in (V_h^{\mathbf{p}})^{G\times N}$ and $\underline{\phi}_p \in (V_{h'}^{\mathbf{p}'})^{G\times N}$
- in particular, if $V_h^{\mathbf{p}}$ is a refinement of $V_{h'}^{\mathbf{p}'}$, it can be written as:

$$\Delta\rho^{hh'} = -\frac{\left\langle \pi_{h'}\underline{\phi}^{\star}_{i,h}, \underline{\phi}_{p,h'} \right\rangle_{\mathcal{B}^{G\times N}_{h'-}} - \left\langle \underline{\phi}^{\star}_{i,h}, \pi_h\underline{\phi}_{p,h'} \right\rangle_{\mathcal{B}^{G\times N}_{h-}}}{\left\langle \underline{\underline{F}}^{\star}_{i,h}\underline{\phi}^{\star}_{i,h}, \underline{\phi}_{p,h'} \right\rangle_{\mathcal{D}^{G\times N}}}$$

- π_h (resp. $\pi_{h'}$), the L^2-projector on $V_h^{\mathbf{p}}$ (resp. $V_{h'}^{\mathbf{p}'}$),
- $\vec{\nabla}_{hh'}$, the broken gradient operator on the union mesh between $\mathcal{M}_h$ and $\mathcal{M}_{h'}$,
- $\left\langle \underline{\varphi}_h, \underline{\phi}_h \right\rangle_{\mathcal{B}^{G\times N}_{h-}} = \sum_{g=1}^{G}\sum_{n=1}^{N} w_n \sum_{\kappa\in\mathcal{M}_h} \left\langle \varphi^{g+}_{h,n}, [\![\phi^{g}_{h,n}]\!] \right\rangle_{\partial\kappa_-}$.

AMR and Perturbation theory

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a "classical" tool in reactor physics to *decompose the variations of a quantity of interest* (e.g. the reactivity $\rho = 1 - \frac{1}{k}$) w.r.t. the *local variations of the parameters* associated with the core configuration (nuclear data, material spatial distribution)

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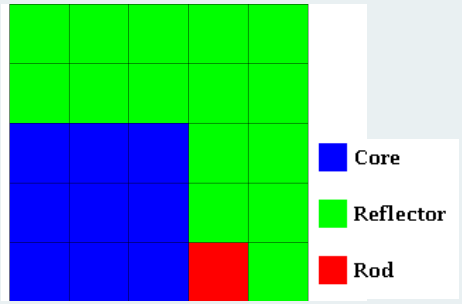
- *a priori*, this extended formula can be practically *useful in two different ways...*

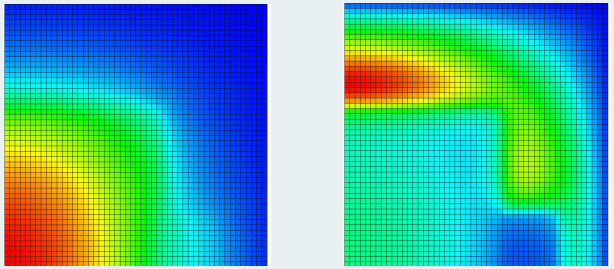
AMR and Perturbation theory

- Evaluating the reactivity effect associated with a modification of the “physical” parameters :

Improve the AMR performance when evaluating a “physical” reactivity change

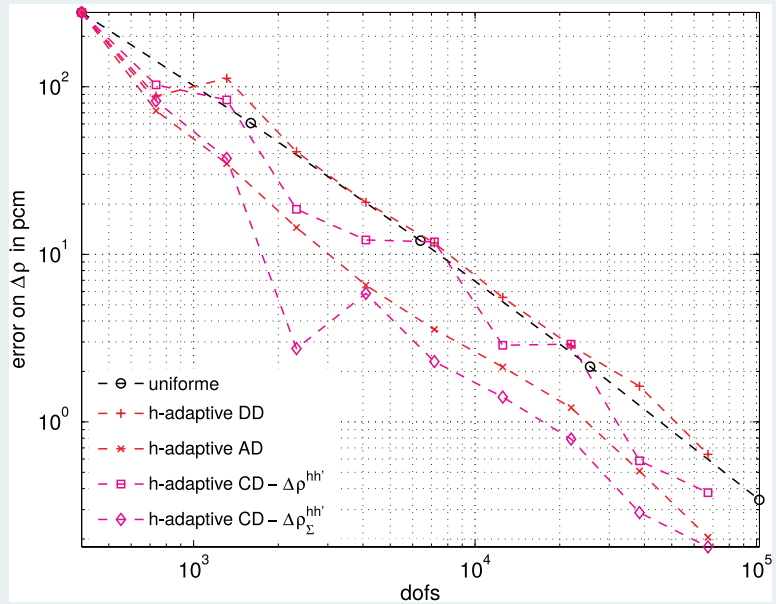


Geometry and core materials



Scalar fluxes – configuration “rod in”

for AMR to be fully interesting when evaluating a reactivity worth, $\Delta\rho_{\Sigma}^{hh'}$ to be used instead of $\Delta\rho^{hh'}$



Error on the rod reactivity “worth” – *h*—refinement

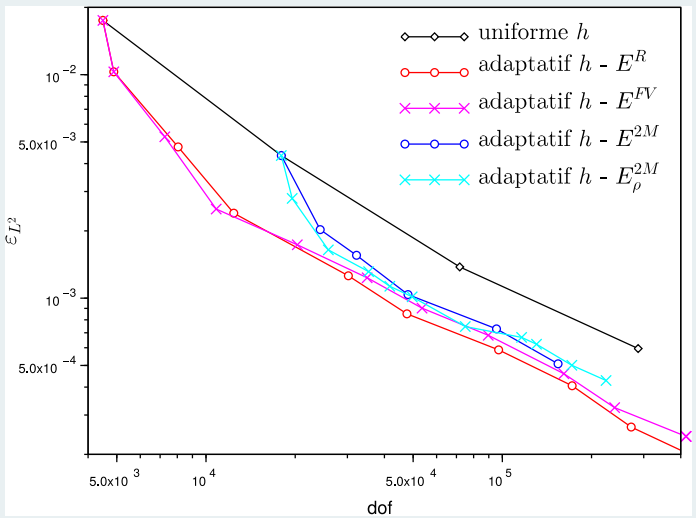
- “direct-driven” (DD) : AMR for direct problem and meshes imposed for adjoint problem
- “adjoint-driven” (AD) : AMR for adjoint problem and meshes imposed for adjoint problem
- “configuration-dependent” (CD) : independent AMR for direct and adjoint problems

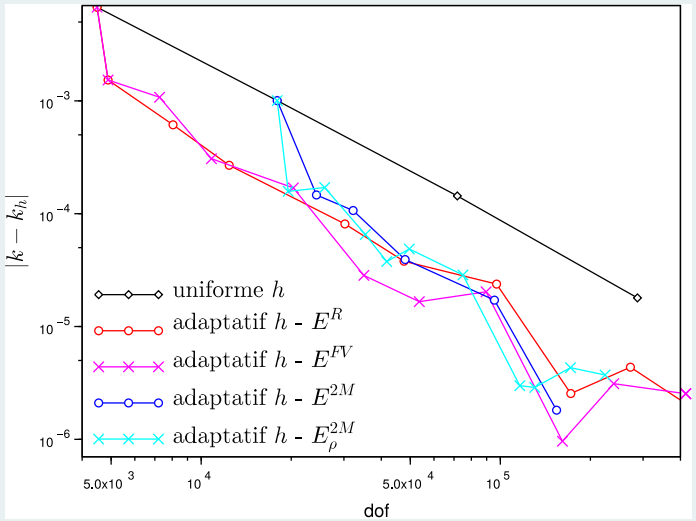
AMR and Perturbation theory

- Evaluating the change in reactivity due to a mesh refinement:

Towards a "ρ-oriented" mesh adaptation?

- *"two-mesh" error estimators*: E^{2M} (flux difference) and E_ρ^{2M} ($\Delta\rho^{hh'}$ decomposition over the mesh)
- *h*-refinement on the ZONA2B benchmark (2D case) with different estimators





- *No significant gain on ρ convergence when using a ρ error estimator vs. a flux error estimator*
  Consistent with the results reported in [Owens et al., 2017] where the more standard "Dual Weighted Residual" approach (firstly used in neutronics in [Lathouwers, 2011]) is used

Summary and perspectives

Summary

- "*Sodium Fast Reactors*" core physics, ASTRID industrial project
- Investigation of *hp—AMR* for reducing the burden of solving the neutron transport (Multigroup, S_N, DG upwing scheme)
- Two different *error estimators/indicators* under very different *regularity hypotheses*
- Importance of the *pre-asymptotic/asymptotic p—convergence regime* on dofs reduction
- Connecting the classical "*perturbation theory*" in reactor core physics with AMR

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And beyond

- AMR with DG upwind on *polygonal meshes*
  (using "non polynomial" element bases $\rightarrow$ D. Labeurthre ongoing Ph.D. thesis with A. Calloo)
- Performance optimization:
  - *Acceleration* of these *nested iterative loops* by a *simplified operator* (e.g. diffusion) based preconditioning (so-called "Diffusion Synthetic acceleration")
  - *Nested iterative loops* for solving the source-flux dependency $\rightarrow$ *where to put the AMR loop?*
- Energy-space coupling:
  - *"Adaptive"* condensed energy mesh for *group-wise spatial mesh adaptation*
  - *Spatial domain decomposition* with *subdomain-dependent energy meshes*

Thank you for your attention

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